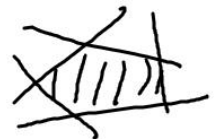


LECTURE 2: CONVEXITY

- A SET $S \subseteq \mathbb{R}^n$ IS CONVEX IF $\left. \begin{matrix} x_1, x_2 \in S \\ \lambda \in (0,1) \end{matrix} \right\} \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S$
- THE INTERSECTION OF CONVEX SETS IS A CONVEX SET
- LET $S = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i=1, \dots, m\}$. THEN IF g_i ARE CONVEX FUNCTIONS, THEN S IS A CONVEX SET.
- THE CONVEX HULL OF A SET S IS THE SET OF ALL CONVEX COMBINATIONS OF POINTS IN S
- POLYTOPE = THE CONVEX HULL OF FINITELY MANY POINTS
- POLYHEDRON = THE INTERSECTION OF FINITELY MANY HALF-SPACES
- REPRESENTATION THM: "POLYHEDRON = POLYTOPE + POLYHEDRAL CONE"



- "EXTREME POINT: A POINT THAT CANNOT BE WRITTEN
TO A CONVEX SET" AS A CONV COMB. OF TWO OTHER POINTS



- A SET $C \subseteq \mathbb{R}^n$ IS A CONE IF $\lambda x \in C$ WHENEVER $x \in C$ AND $\lambda > 0$



- SEPARATION THM:



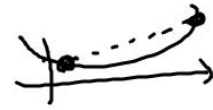
- FARKAS' LEMMA: EXACTLY ONE OF THE SYSTEMS

$$\begin{cases} Ax = b \\ x \geq 0 \end{cases}$$

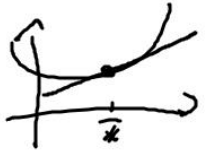
$$\begin{cases} A^T y \leq 0 \\ b^T y > 0 \end{cases}$$

↑ VERIFY INEQ :)

CONVEX FUNCTIONS



- A function f is convex if $f(\lambda x^1 + (1-\lambda)x^2) \leq \lambda f(x^1) + (1-\lambda)f(x^2)$
- A function f is concave if $-f$ is convex
- THE SUM OF CONVEX FUNCTIONS IS A CONVEX FUNCTION.
- THE EPIGRAPH OF A FUNCTION f IS CONVEX $\Leftrightarrow f$ IS CONVEX
- IF $f \in C^1$: THEN f IS CONVEX IF $f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$
- IF $f \in C^2$: ———— IF $\nabla^2 f(x) \succeq 0$ FOR ALL x



CONVEX PROBLEM

- A PROBLEM $\begin{cases} \min f(x) \\ \text{s.t. } x \in S \end{cases}$ IS A CONVEX PROBLEM IF f IS CONVEX ON S AND S IS CONVEX

LECTURE 3: OPTIMALITY CONDITIONS

CONSIDER (P) $\min f(x)$
s.t. $x \in S$

- GLOBAL MINIMA -
- LOCAL MINIMA -
- FUNDAMENTAL THM OF OPTIMALITY: IF (P) IS A CONVEX PROBLEM
THEN EVERY LOCAL MIN IS A
GLOBAL MIN.
- WEIERSTRASS' THEOREM:
 - S NONEMPTY, CLOSED
 - f WEAKLY COERCIVE W.R.T. S
 - f LOWER SEM-CONT $\} \Rightarrow \exists$ GLOBAL MIN
TO (P).

UNCONSTRAINED PROBLEMS

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}^n \end{array}$$

- IF $f \in C^1$: x^* LOCAL MIN $\Rightarrow \nabla f(x^*) = 0$
- IF $f \in C^2$: $\text{--- " ---} \Rightarrow \begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \succeq 0 \end{cases}$
- IF f IS CONVEX: x^* GLOBAL MIN $\Leftrightarrow \nabla f(x^*) = 0$

[PROOF IS SIMPLE
AND USEFUL]

CONSTRAINED PROBLEMS

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

- IF x^* IS A LOCAL MIN $\Rightarrow$

EMILS OPT. COND.
"IT SHOULD NOT BE POSSIBLE TO FIND A DIRECTION AT x^* WHICH IS BOTH A FEASIBLE AND A DESCENT DIRECTION".

$$\Rightarrow \nabla f(x^*)^T p \geq 0 \quad \text{FOR ALL FEASIBLE DIRECTIONS } p.$$

- IF S IS A CONVEX SET.

x^* LOCAL MIN $\Rightarrow x^*$ STATIONARY

$$a) \nabla f(x^*)^T (x - x^*) \geq 0 \quad \forall x \in S$$

$$b) \min_{x \in S} \nabla f(x^*)^T (x - x^*) = 0$$

$$c) x^* = \text{Proj}_S (x^* - \nabla f(x^*))$$

$$d) -\nabla f(x^*) \in N_S(x^*)$$

LECTURE 4: UNCONSTRAINED OPT. METHODS

- LINE SEARCH TYPE ALGORITHMS

$\left\{ \begin{array}{l} 0 \text{ INITIALIZE} \\ 1 \text{ FIND DESCENT DIR} \\ 2 \text{ FIND STEP LENGTH} \\ 3 \text{ UPDATE} \end{array} \right.$

- SEARCH DIRECTION

- STEEPEST DESCENT:

$$p_k = -\nabla f(x_k)$$

- NEWTON

$$\nabla^2 f(x_k) p_k = -\nabla f(x_k)$$

- LEWENBERG-MARGUAROT

$$[\nabla^2 f(x_k) + \gamma I] p_k = -\nabla f(x_k)$$

- STEP LENGTHS

- ARMIJO'S RULE

- EXACT

LECTURE 5/6: OPTIMALITY CONDITIONS

GEOMETRIC OPTIMALITY CONDITIONS:

$$\begin{array}{ll} \min f(x) \\ \text{s.t. } x \in S \end{array}$$

- EMILS OPT COND:

x^* LOCAL MIN $\Rightarrow$ "SHOULD NOT EXIST VECTORS p WHICH ARE BOTH DESCENT/FEASIBLE"

$$\Rightarrow A(x^*) \cap B(x^*) = \emptyset$$

$\uparrow$
FEASIBLE
DIRECTIONS

$\uparrow$
DESCENT
DIRECTIONS

$\downarrow$
TANGENT CONE

$$T_S(x) = \left\{ p \mid \exists \{x_k\} \subseteq S, \{\lambda_k\} \geq 0: \lim x_k = x, \lim \lambda_k(x_k - x) = p \right\}$$

$$\downarrow \quad \bar{F}(x) = \{ p : \nabla f(x)^T p < 0 \}$$

GEOMETRIC OPT COND:

$$x^* \text{ LOCAL MIN} \Rightarrow \dot{F}(x^*) \cap T_S(x^*) = \emptyset$$

KKT CONDITIONS

- ASSUME $S = \{x \mid g_i(x) \leq 0, i=1, \dots, m\}$
- GRADIENT CONE: $G(x) = \{p \mid \nabla g_i(x)^T p \leq 0, i \in I(x)\}$
- $T_S(x) \subseteq G(x)$
- ABADIE'S CQ: $T_S(x) = G(x)$
- ASSUMING ABADIE'S CQ $\Rightarrow x^* \text{ LOCAL MIN} \Rightarrow \dot{F}(x^*) \cap G(x^*) = \emptyset$

$$\Rightarrow x^* \text{ LOCAL MIN} \Rightarrow \dot{F}(x^*) \cap G(x^*) = \emptyset$$

- THIS CAN BE FORMULATED AS AN INCONSISTENT INEQ SYSTEM

$$x^* \text{ LOCAL MIN} \Rightarrow \begin{cases} \nabla f(x^*)^T p < 0 \\ \nabla g_i(x^*)^T p \leq 0, \quad i \in I(x^*) \end{cases} \quad \text{INCONSISTENT}$$

$$\Rightarrow [\text{FARKAS' LEMMA}]$$

$$\Rightarrow \left\{ \begin{array}{l} \nabla f(x^*) + \sum_{i=1}^m \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0, \quad i=1, \dots, m \\ g_i(x^*) \leq 0, \quad i=1, \dots, m \\ \mu_i \geq 0, \quad i=1, \dots, m \end{array} \right. \quad \underline{\text{SOLVABLE}}$$

KKT SYSTEM

- IF ABADIE'S CG HOLDS :

[NECESSARY] $\vec{x}^*$ LOCAL MIN $\Rightarrow \vec{x}^*$ KKT POINT

- IF THE PROBLEM IS CONVEX

[SUFFICIENT] $\vec{x}^*$ KKT POINT $\Rightarrow \vec{x}^*$ GLOBAL MIN

LECTURE 7:

LAGRANGIAN DUALITY.

- $f^* = \inf_{x \in \mathcal{X}} f(x)$
s.t. $g_i(x) \leq 0, i=1, \dots, m$
- g_i - CONSTRAINTS ARE COMPLICATED $\Rightarrow$ LIFT THEM TO THE OBJECTIVE WITH A PENALTY
- LAGRANGIAN DUAL FUNCTION: $q(\mu) = \min_{x \in \mathcal{X}} f(x) + \mu^T g(x)$ ALWAYS CONCAVE DUAL PROBLEM
 $q^* = \sup_{\mu \geq 0} q(\mu)$
s.t. $\mu \geq 0$
- WEAK DUALITY: FOR ANY FEASIBLE x AND ANY $\mu \geq 0$:
 $q(\mu) \leq f(x)$
- STRONG DUALITY: IF SLATER'S CQ HOLDS AND f IS CONVEX, THEN
 $f^* = q^*$

LECTURE 8/9 LINEAR PROGRAMMING

- MINIMIZE A LINEAR FUNCTION OVER A POLYHEDRON
- IF AN OPTIMAL SOLUTION EXISTS $\Rightarrow$ ONE OF THE EXTREME POINTS MUST BE OPTIMAL
- STANDARD FORM:
$$\begin{cases} \min C^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{cases}$$
- EXTREME POINTS $\Leftrightarrow$ BASIC FEASIBLE SOLUTIONS (BFS)
- BFS: PARTITION $A = [B, N]$ SUCH THAT $\text{rank } B = m$ AND $B^{-1}b \geq 0$
- DEGENERATE BFS: $B^{-1}b \not\geq 0$ 
- SIMPLEX METHOD: "ITERATIVELY UPDATES PARTITION $A = [B, N]$ BY REPLACING ONE COLUMN IN B WITH A COLUMN FROM N "

- PHASE I - SIMPLEX

→ FIND INITIAL BFS.

$$(P) \begin{cases} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$$

INTRODUCE
var. a_i

$\Rightarrow$

$$(P') \begin{cases} \min & 1^T a_i \\ \text{s.t.} & Ax + a_i = b \\ & x \geq 0 \\ & a_i \geq 0 \end{cases}$$

$$\sum_{i=1}^m a_i$$

$$\begin{aligned} \tilde{A} &= [A, I] \\ \tilde{A} \begin{bmatrix} x \\ a_i \end{bmatrix} &= b \end{aligned}$$

— SOLVE (P') WITH SIMPLEX METHOD.

— THIS CAN BE DONE SINCE USING $x_B = a_i$ IS A BFS
 $\Rightarrow A = [I, A]$

— WHEN OPTIMAL SOLUTION FOR (P') IS FOUND, NO
ARTIFICIAL VARIABLES IS INCLUDED IN THE BASIC VARIABLES

— YOU NOW HAVE AN INITIAL BFS TO (P)

LECTURE 10 LP DUALITY

$$\begin{array}{lcl}
 \text{PRIMAL} & & \text{DUAL} \\
 - \min c^T x & \Leftrightarrow & \max b^T y \\
 \text{s.t. } Ax = b & & \text{s.t. } A^T y \leq c \\
 x \geq 0 & & y \in \mathbb{R}^m \text{ (FREE)}
 \end{array}$$

n VARIABLES m VARIABLES
 m CONSTRAINTS n CONSTRAINTS

- WEAK DUALITY: FOR x AND y FEASIBLE IN THEIR PROBLEMS

$$c^T x \geq b^T y$$

- STRONG DUALITY: IF BOTH PROBLEMS FEASIBLE, THEN

$$c^T x^* = b^T y^*$$

- COMPLEMENTARY SLACKNESS: " EITHER THE SLACK OR THE DUAL VARIABLE CORRESPONDING TO A CONSTRAINT IS 0 "

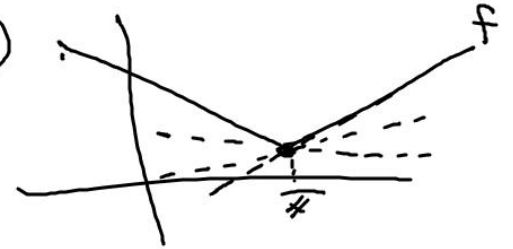
- MEANING OF OPTIMAL DUAL VARIABLES:

u_i^* IS HOW MUCH WILL THE OPTIMAL OBJECTIVE VALUE OF THE PRIMAL PROBLEM CHANGE IF b_i IS INCREASED WITH 1.

LECTURE 11: CONVEX OPTIMIZATION

- A SUBGRADIENT p TO A CONVEX FUNCTION f AT $\bar{x}$ IS A VECTOR FULFILLING:

$$f(x) \geq f(\bar{x}) + p^T(x - \bar{x})$$



- THEY ALWAYS EXIST FOR CONVEX FUNCTIONS
- IF f IS DIFFERENTIABLE AT $\bar{x}$, THEN

$$\partial f(\bar{x}) = \{ \nabla f(\bar{x}) \}$$

- SUBGRADIENT METHOD:

$$x_{k+1} = x_k - \alpha_k p_k$$

SUBGRADIENT



LECTURE 3

FEASIBLE DIRECTION METHODS

- ITERATIVE METHODS
 - 0: INITIALIZE
 - 1: FIND SEARCH DIRECTION WHICH IS FEASIBLE
 - 2: STEP LENGTH
 - 3: UPDATE
- FRANK-WOLFE : LINEARIZE PROBLEM IN EACH ITERATION AND SOLVE LP.
[S POLYHEDRON]
- SIMPLICIAL DECOMPOSITION : SIMILAR TO F-W. BUT REMEMBERS VISITED EXTREME POINTS
- GRADIENT PROJECTION ALG :
[S EASY TO PROJECT TO] $x_{k+1} = \text{Proj}_S [x_k - \alpha_k \nabla f(x_k)]$

LECTURE 14

CONSTRAINED OPT. METHODS

$$\left. \begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array} \right\} \Rightarrow \min_{x \in \mathbb{R}^n} f(x) + \chi_S(x) \quad \chi_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise} \end{cases}$$

- EXTERIOR PENALTY METHOD [PENALIZE BEING INFEASIBLE]
 - APPROXIMATE INDICATOR FUNCTION FROM BELOW $\chi_S^v(x) \leq \chi_S(x)$
- INTERIOR PENALTY METHOD [PENALIZE BEING CLOSE TO CONSTRAINTS]
 - APPROXIMATE $\chi_S(x)$ FROM ABOVE $\hat{\chi}_S(x) \geq \chi_S(x)$

TIPS:

- TAKE IT EASY! DON'T DO THE PROBLEMS IN NUMERICAL ORDER.
- DRAW!
- ALWAYS CHECK FEASIBILITY IN METHODS
- ONLINE (DIGITAL)
 - EXAM IS BASICALLY THE SAME
 - THEORY-LIST WON'T BE TESTED
 - ALL AIDS ALLOWED
 - MORE INFORMATION WILL COME!

$$\begin{array}{l} \min f(x) \\ \text{s.t. } x \in X \end{array} \Leftrightarrow \begin{array}{l} \max -f(x) \\ \text{s.t. } x \in X \end{array}$$