Examples of Problem Solving Exam Exercises LMA017

In all exercises, it is of high importance to justify your answers! Just giving the correct answer will normally result in zero points.

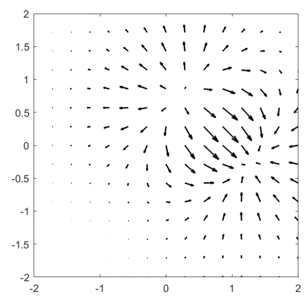
Most of the exercises are in the 'red zone' when it comes to difficulty that is acceptable on an exam. Some of them are well above it – 4,13 and 14 are definitely in this category. Less than a third of the exam will be problems of this type! More exact info can be retrieved from the exercise exam, which will be posted later.

Solutions will be posted later.

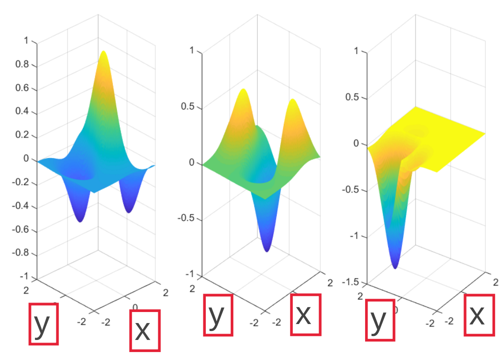
Week 1

1. Functions of several variables, Visualization

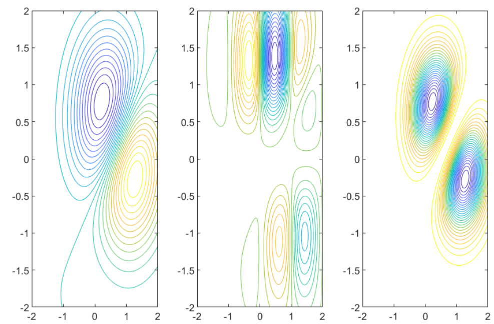
The following plot depicts the gradient ∇f of a function $f : \mathbb{R}^2 \to \mathbb{R}$.



(a) Which one of following graphs shows $\frac{\partial f}{\partial x}$?



(b) Which one of the following contour plots depicts f?



2. Limits and continuity

Consider the following function

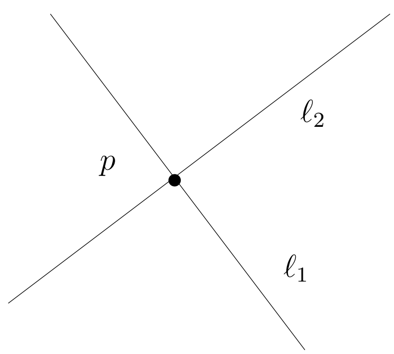
$$f(x, y) = \begin{cases} 0 & \text{if } y = x^3 \\ 1 & \text{else.} \end{cases}$$

Determine the points in which f is continuous.

Week 2

3. Derivatives I

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has the following property: There exists two non-parallell lines ℓ_1 and ℓ_2 on which f is constantly equal to 1. Let $\mathbf{p}$ be the intersection of the two lines.



(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$ be the directional vectors of the lines. Show that $D_{\mathbf{v}_1} f(\mathbf{p}) = D_{\mathbf{v}_2} f(\mathbf{p}) = 0$.

(b) Assume that f is differentiable. Show that $\nabla f(\mathbf{p}) = \mathbf{0}$.

(c) Does f have to be differentiable? Show this, or give a counterexample. If you give a counterexample, make sure to specify the lines ℓ_1 and ℓ_2.

4. Derivatives II

Two hiking groups are moving through a landscape. The height of the landscape is described by a differentiable function h. The trajectories of the two groups are described by

$$\gamma_1(t) =< t, t^2 >, \text{ and } \quad \gamma_2 =< t, 3t >, \text{ respectively.}$$

Time is measured in minutes. The two groups record how their heights are changing with time. At the time $t = 0$ min, when the two groups meet in the point $< 0, 0, h(0, 0) >$, group 1 reports that their height is rising with a rate 0.1 m/min and group 2 reports that their height is rising with 0.4 m/min.

1. Use the chain rule to write down a formula involving $h(0, 0)$, $\nabla h(0, 0)$, $\gamma_1(0)$, and $\gamma_1'(0)$ that expresses the rate of height change the first hiking group is experiencing at $t = 0$.

2. What is the gradient of h in $(0, 0)$?

5. The Schrankensatz

Two drones are moving through the air. Their positions relative to a base station are determined by a gps. They are both equipped with altimeters, which determine their heights above sea level. At a certain times, the measurement reads as follows

- Drone 1: Position $< 200, 400 >$, height 35.
- Drone 2: Position $< -80, 130 >$, height 50.

All readings are in meters. We can assume that the errors of the gps measurements of each coordinate is not larger than 5 m, and that the height measurements are off by at most 1 meter.

Use the Schrankensatz to determine the (three-dimensional) distance between the drones, with error bounds.

Week 3

6. Optimization I

Consider the function

$$g : \mathbb{R}^2 \to \mathbb{R}, < x, y > \mapsto (x^2 + y^2)e^{-x^2-y^2}$$

(a) Determine and classify all stationary points of g.

(b) Does g have a global maximum? A global minimum?

7. Optimization II

Assume that we are given a fixed amount of batter to make an ice-cream cone. How should we design the cone to fit as much icecream as possible in the cone?

8. Optimization III

Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function.

(a) Show that g has a maximum on the unit sphere.

(b) Show that there must exist a point $\mathbf{p}$ on the sphere in which $\nabla g(\mathbf{p})$ is parallel to $\mathbf{p}$.

9. Optimization IV

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose that both functions

$$g(t) = f(t, 0) \text{ and } h(t) = f(0, t)$$

have maximums in $t = 0$. Does f need to have a maximum in $(0, 0)$? Prove it or give a counterexample.

Week 4

10. Integrals and probability

We choose a point $\mathbf{p} =< x, y, z >$ uniformly at random in the unit cube $[0, 1]^3$. What is the probability that z is larger than $x^2 + y^2$?

11. Integrals

The unit circle is filled with a material whose density is varying according to

$$\rho(x, y) = \frac{1}{1 + 0.8 \cdot (x^2 + y^2) - 0.1 \cdot \cos(x)y + 0.1 \cdot y\cos(y)}$$

Show that the circle weighs more than

$$\frac{2\pi}{0.8} \cdot \ln(2) \text{ mass units..}$$

12. Variable substitution I

Determine the area of the domain in the plane which is bounded by the curves $y = 2/x, y = 1/x$, $y^2 = x^2 + 1$ and $y^2 = x^2 + 2$. *Tip:* Use variable substitution!

Week 5

13. Variable substitution II

A positive function which is continuous everywhere on the unit ball except the origin is called *integrable* if the limit

$$\lim_{\epsilon \to 0+} \iiint_{C_\epsilon} f \, \mathrm{d}V < \infty.$$

Here, $C_\epsilon = \{\mathbf{p} \,|\, \epsilon \leq |\mathbf{p}| \leq 1\}$.

For which α is the function

$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)\alpha}$$

integrable over the unit ball?

14. Variable substitution III

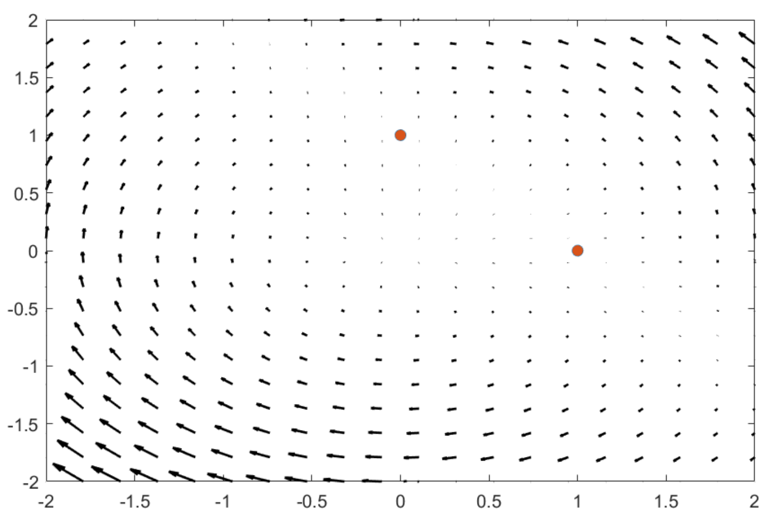
Determine the integral of

$$f(x,y,z) = \cos(x+y+z)$$

over the unit ball $\{\mathbf{p} \,|\, |p| \leq 1\}$. *Tip:* Rotate the problem!

15. Curve integrals

Below, a quiver plot of a vector field is shown. The two points $< 1, 0 >$ and $< 0, 1 >$ are marked. The vector field is zero on the line $x + y = 1$, and defined everywhere in $\mathbb{R}^2$.



(a) Describe a curve γ_2 between $< 1, 0 >$ and $< 0, 1 >$ for which

$$\int_{\gamma_1} \mathbf{v} \cdot d\mathbf{r} = 0.$$

(b) Sketch a curve γ_2 between $< 1, 0 >$ and $< 0, 1$ for which

$$\int_{\gamma_2} \mathbf{v} \cdot d\mathbf{r} > 0.$$

(c) Can the vector field satisfy the differential equation

$$\frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial y}?$$

Week 6 - 8

16. Flow integrals I

Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ a differentiable vector field. Assume that

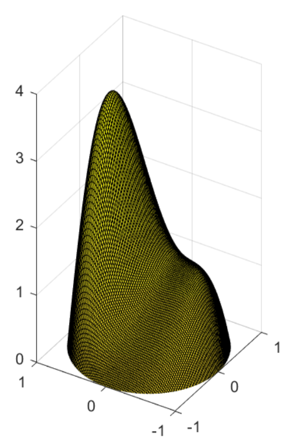
$$\mathbf{v}(\mathbf{p}) \cdot \mathbf{p} < 0$$

for all $\mathbf{p} \in \mathbb{R}^3$. Can it be that $\operatorname{div}(\mathbf{v}) = 0$ in all of $\mathbb{R}^3$? Give an example of such a vector field, or prove that it cannot exist. *Tip*: Investigate the flow integral of $\mathbf{v}$ over the unit sphere.

17. Flow integrals II

Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector fields

$$\mathbf{v}(x, y, z) = (x^2 + y^2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}(x, y, z) = \begin{bmatrix} -yx^2 \\ xy^2 \\ 0 \end{bmatrix}.$$



1. Show that $\mathbf{v} = \operatorname{curl}\mathbf{w}$.
2. Let K be the part of the graph of the function

   $$g(x, y) = (1 - x^2 - y^2)e^{x+y}(1 + x^2)(1 + 4y^2)$$

   which lies above the xy-plane (it is the surface shown in the figure). Calculate

   $$\int_K \mathbf{v} \cdot d\mathbf{S}$$