Tentamen MVE255

Mathematical sciences, Chalmers University of Technology.

Date:	–
Examiner:	Axel Flinth
Allowed aids:	Any.
Grade limits:	20 points for the grade 3. 30 points for the grade 4 40 points for the grade 5.

There are in total 50 points to collect.

Calculations and arguments should be presented in full. **Only providing an answer will normally not be rewarded with points.** Solutions may be written in Swedish or English (or German).

If you use any external tool or resource, you should reference it. See the Canvas Page for more information. Not referencing properly may result in point deduction.

The exam consists of **eight (8)** problems. They are distributed over **four (4)** sheets.

Good luck!

PRACTICE EXAM

Solutions will be posted on October 19th.

SOLUTIONS

Problem 1 (6p)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined through

$$f(x,y) = xe^{xy}, \quad \langle x,y\rangle \in \mathbb{R}^2.$$

(a) Calculate the gradient ∇f of f. (2p)

(b) Calculate the Hessian f'' of f. (2p)

(c) Evaluate f, ∇f and f'' in $< 0, 0 >$. (2p)

Solution. (a): The partial derivatives of f are given by

$$\frac{\partial f}{\partial x}(x,y) = e^{xy} + xye^{xy}, \qquad \frac{\partial f}{\partial y}(x,y) = x^2 e^{xy}.$$

Therefore,

$$\nabla f(x,y) = \begin{bmatrix} e^{xy}(1+xy) \\ x^2 e^{xy} \end{bmatrix}.$$

(b): We calculate the partial derivatives of ∇f

$$\frac{\partial}{\partial x}\nabla f(x,y) = \begin{bmatrix} e^{xy}y(1+xy) + ye^{xy} \\ 2xe^{xy} + yx^2e^{xy} \end{bmatrix}, \qquad \frac{\partial}{\partial y}\nabla f(x,y) = \begin{bmatrix} xe^{xy} + xe^{xy} + x^2ye^{xy} \\ x^3e^{xy} \end{bmatrix}.$$

Thus,

$$f''(x,y) = \begin{bmatrix} ye^{xy}(2+xy) & (2x+yx^2)e^{xy} \\ (2x+yx^2)e^{xy} & x^3e^{xy} \end{bmatrix}.$$

(c): We set $x = y = 0$ in the three expressions to obtain

$$f(0,0) = 0, \quad \nabla f(0,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f''(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

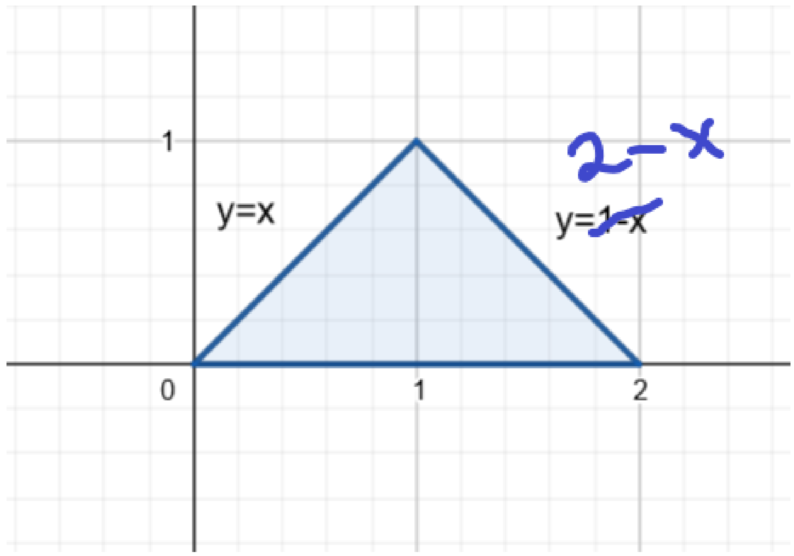
Problem 2 **(7p)**

Calculate $\iint_T xy \, \mathrm{d}x\mathrm{d}y$, where T is the triangle with corners $< 0, 0 >$, $< 1, 1 >$ and $< 2, 0 >$.

Solution. The triangle is described by the equations

$$0 \le x \le 1, 0 \le y \le x \text{ or } 1 \le x \le 2, 0 \le y \le 2 - x,$$

as can be seen in the sketch below.



The theorem of Fubini therefore implies

$$
\begin{aligned}
\iint_T xy \,\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^x xy \,\mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{2-x} xy \,\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left[\frac{xy^2}{2}\right]_{y=0}^{y=x} \mathrm{d}x + \int_1^2 \left[\frac{xy^2}{2}\right]_{y=0}^{y=2-x} \mathrm{d}x \\
&= \int_0^1 \frac{x^3}{2}\,\mathrm{d}x + \int_1^2 \frac{x(2-x)^2}{2}\,\mathrm{d}x = \int_0^1 \frac{x^3}{2} + \frac{(1+x)(1-x)^2}{2}\,\mathrm{d}x \\
&= \left[\frac{x^4}{4} - \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{2}\right]_0^1 = \frac{1}{3}.
\end{aligned}
$$

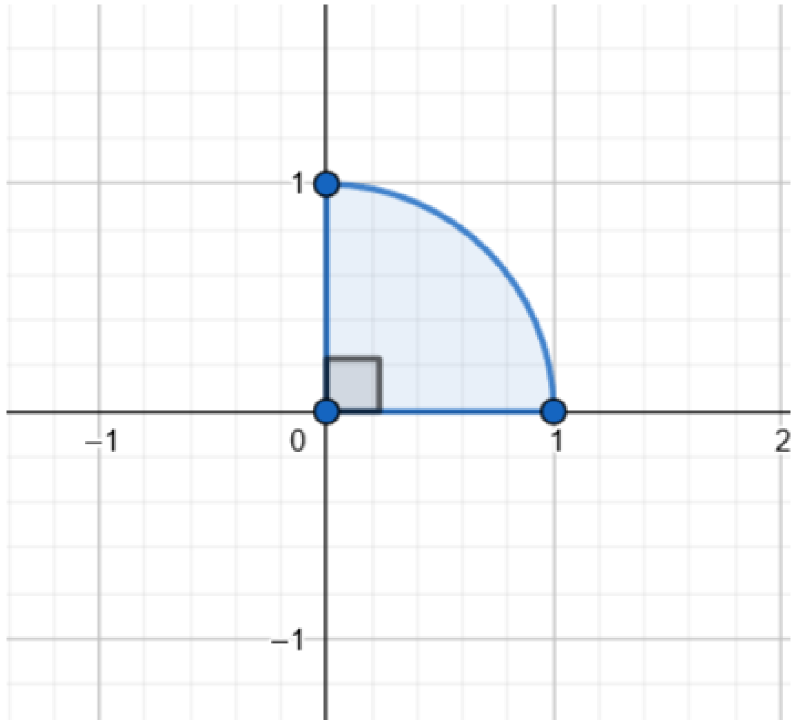
In the second line, we substituted $\tilde{x} = x - 1$ to get a single integral to evaluate in the final line. □

Problem 3 **(6p)**

Calculate the center of mass of the part of the unit disk which lies in the region $\{\langle x, y\rangle \mid x, y \geq 0\}$.

Solution. The region (let's denote it with Q) is a quarter of a circle of radius 1. Consequently, its area is given by $\pi/4$.

To calculate the center of mass, we now need to integrate x and y over the region. The region is simple to describe in polar coordinates - the r-values range from 0 to 1, and θ from 0 to $\pi/2$ (for each r).



We may thus transform the integrals to polar coordinates.

$$\iint_Q x \,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{\pi/2} (r\cos(\theta))r \,\mathrm{d}\theta\mathrm{d}r$$
$$\iint_Q y \,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{\pi/2} (r\sin(\theta))r \,\mathrm{d}\theta\mathrm{d}r$$

Notice that r is the functional determinant of polar coordinates. We now evaluate the integrals

$$\int_0^1 \int_0^{\pi/2} (r\cos(\theta))r \,\mathrm{d}\theta\mathrm{d}r = \int_0^1 [r^2\sin(\theta)]_{\theta=0}^{\theta=\pi/2} \,\mathrm{d}r = \int_0^1 r^2 \,\mathrm{d}r = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3}$$
$$\int_0^1 \int_0^{\pi/2} (r\sin(\theta))r \,\mathrm{d}\theta\mathrm{d}r = \int_0^1 [-r^2\cos(\theta)]_{\theta=0}^{\theta=\pi/2} \,\mathrm{d}r = \int_0^1 -r^2(-1) \,\mathrm{d}r = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3}.$$

The center of gravity is thus given by

$$\frac{1}{\iint_Q 1 \,\mathrm{d}x\mathrm{d}y} \left\langle \iint_Q x \,\mathrm{d}x\mathrm{d}y, \iint_Q y \,\mathrm{d}x\mathrm{d}y \right\rangle = \left\langle \frac{4}{3\pi}, \frac{4}{3\pi} \right\rangle.$$

$\square$

Problem 4 **(7p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined through

$$f(x, y) = xe^y.$$

The optimization problem

$$\max f(x, y) \text{ subject to } x + y = 1$$

has a unique solution (this does not need to be shown).

(a) Find the solution. (4p)

(b) Does the corresponding minimization problem

$$\min f(x, y) \text{ subject to } x + y = 1$$

have a solution? (3p)

Solution. (a) The solution $\langle x, y\rangle$ of the problem is also a local maximum of the constrained optimization problem. As such, the Lagrange criterion must be true in that point, given that the gradient of the constraint function is not zero there. Denoting $g(x, y) = x + y - 1 = 0$, we however have

$$\nabla g = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which surely never vanishes. We must thus have

$$\nabla f(x, y) = \begin{bmatrix} e^y \\ xe^y \end{bmatrix} = \lambda \nabla g(x, y) = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$$

for some $\lambda \in \mathbb{R}$. This means that $e^y = \lambda = xe^y$, which can only be true of $x = 1$. The constraint then shows us that $y = 1 - 1 = 0$.

Since we were given the fact that the problem has a solution, the point we just have found must be the point where the maximum is attained. The solution is thus $\langle 1, 0\rangle$.

(b) For $t \in \mathbb{R}$ arbitrary, $\langle t, 1 - t\rangle$ is feasible. Since

$$f(t, 1 - t) = te^{1-t} \to -\infty \cdot \infty = -\infty$$

when $t \to \infty$, the problem cannot have a minimum.

We can alternatively argue that if the problem would have a minimum, it would have to be attained in a point where the Lagrange criterion is true, i.e. the point we found in (a). If the minimum is equal to the maximum, f would have to be constant on the set of feasible points, which clearly is not the case ($f(0, 1) = 0 \neq f(1, 0) = e$). Thus, the problem cannot have a minimum. $\square$

Problem 5 **(6p)**

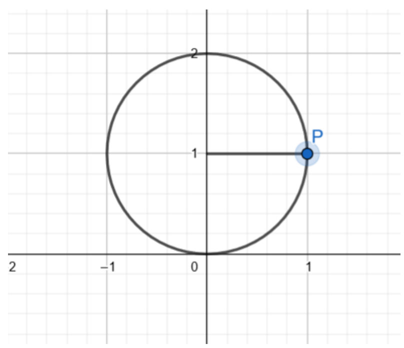
A circle has a known center in $\langle 0, 1 \rangle$. In order to determine it fully, we measure the position of one point $\mathbf{p}$ on its perimiter. The measurement reads

$$\mathbf{p} = < 1, 1 > .$$

Each coordinate of the measurement has an error, which is smaller than 0.1 length units. Use the Schrankensatz to give an estimate of the area of the circle, with error bounds.

Solution. The radius of the circle is given by the formula

$$R(\mathbf{p}) = R(x, y) = |\langle x, y \rangle - \langle 0, 1 \rangle| = \sqrt{x^2 + (y-1)^2}.$$



We quickly calculate $R(1, 1) = 1$. If we denote the actual position of the point $\tilde{x}, \tilde{y}$, our task is now to give a bound for $|R(\tilde{x}, \tilde{y}) - R(0, 1)|$. The Schrankensatz tells us that

$$|R(\tilde{x}, \tilde{y}) - R(1, 1)| \leq M_1 |\tilde{x} - 1| + M_2 \tilde{y} - 1 \leq 0.1M_1 + 0.1M_2,$$

where the final inequality is given by the premise of the problem, and

$$M_i = \max_{|x-1| \leq 0.1, |y-1| \leq 0.1} \frac{\partial R}{\partial p_i}(x, y)$$

We have

$$\frac{\partial R}{\partial x} = \frac{x}{\sqrt{x^2 + (y-1)^2}}, \quad \frac{\partial R}{\partial y} = \frac{y-1}{\sqrt{x^2 + (y-1)^2}}.$$

We now estimate M_i by choosing x and y so that the nominator is as large as possible, and the denominator as small as possible:

$$M_1 \leq \frac{1.1}{\sqrt{0.9^2 + 0^2}} = \frac{1.1}{0.9}, \quad M_2 \leq \frac{0.1}{\sqrt{0.9^2 + 0^2}} = \frac{0.1}{0.9}$$

Note that in order to get $(y-1)^2$ as small as possible among all values of $y \in [0.9, 1.1]$, we need to choose $y = 1$. We conclude

$$|R(\tilde{x}, \tilde{y}) - R(1, 1)| \leq 0.1M_1 + 0.1M_2 \leq \frac{1.1 \cdot 0.1}{0.9} + \frac{0.1^2}{0.9} = \frac{1.2 \cdot 0.1}{0.9},$$

i.e.

$$R(\tilde{x}, \tilde{y}) = 1 \pm \frac{12}{90}.$$

□

Problem 6 **(6p)**

Let $\alpha \in \mathbb{R}$ be a parameter. Consider the curves given by the parametrisations

$$\gamma_\alpha(t) = \begin{bmatrix} t\cos(\pi\alpha t) \\ t\sin(\pi\alpha t) \\ t^2(1 + 2\sin^2(t)) \end{bmatrix}, \quad t \in [0, 1]$$

and the vector field

$$\mathbf{v}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ z\cos(z) + \sin(z) \end{bmatrix}.$$

(a) Prove that $\mathbf{v}$ is conservative, and calculate a potential of $\mathbf{v}$. (3p)

(b) Prove that the curve integral of $\mathbf{v}$ is the same along all the curves. I.e. prove that

$$\int_{\gamma_\alpha} \mathbf{v} \cdot d\mathbf{r}$$

does not depend on α. (3p)

Solution. (a) We make an ansatz $g(x, y, z)$ for the potential:

$$\frac{\partial g}{\partial x} = 0$$
$$\frac{\partial g}{\partial y} = 0$$
$$\frac{\partial g}{\partial z} = z\cos(z) + \sin(z).$$

The first two equations here imply that g only depends on z, $g = g(z)$. The final line then implies

$$g = \int z\cos(z) + \sin(z)\,\mathrm{d}z = [\text{partial integration}] = z\sin(z) + \int -\sin(z) + \sin(z)\,\mathrm{d}z = z\sin(z) + C.$$

Thus, $g(z) = z\sin(z)$ is a potential of $\mathbf{v}$. *Note that we do not need to check the integrability criterion, we can indeed just calculate a potential.*

(b) The fundamental theorem of line integrals imply that

$$\int_{\gamma_\alpha} \mathbf{v} \cdot d\mathbf{r} = g(\gamma_\alpha(1)) - g(\gamma_\alpha(0)).$$

We have

$$\gamma_\alpha(0) = \langle 0, 0, 0 \rangle, \quad \gamma_\alpha(1) = \big\langle \cos(\pi\alpha), \sin(\pi\alpha), (1 + 2\sin^2(1) \big\rangle$$

Thus, all start points and all endpoints share the same z-coordinate. Since the potential only depends on z, this means that all integrals share a common value, which was what to be shown. □

Problem 7 **(6p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x, y) = \begin{cases} y & \text{if } y = x^2 \\ 0 & \text{else.} \end{cases}$$

(a) Is f partially differentiable in $\langle 0, 0 \rangle$? (3p)

(b) Is f continuous in $\langle 0, 0 \rangle$? (3p)

Solution. (a) We can use the definition of the partial derivatives to calculate them:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t}, \quad \frac{\partial f}{\partial y}(0, 0) = \lim_{t \to 0} \frac{f(0, t) - f(0, 0)}{t}$$

Since $0 \neq t^2$ and $t \neq 0^2$ for any $t \neq 0$, we have $f(t, 0) = f(0, t) = 0$ for all t. We also have $f(0, 0) = 0$ ($\langle 0, 0 \rangle$ is a point in which $y = x^2$, but $y = 0$ in this point). Therefore

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{0}{t} = 0, \quad \frac{\partial f}{\partial y}(0, 0) = \lim_{t \to 0} \frac{0}{t} = 0.$$

The function is thus partially differentiable in $\langle 0, 0 \rangle$.

(b) For each point $\langle x, y \rangle$, we have $|f(x, y)| \leq |y|$ (in points where $y = x^2$, we have equality, and in the other points, the inequality $|f(x, y)| = 0 \leq |y|$ is also clear. Since $|y|$ clearly tends to 0 as $\langle x, y \rangle \to \langle 0, 0 \rangle$, the squeeze theorem proves that

$$\lim_{\langle x, y \rangle \to \langle 0, 0 \rangle} f(x, y) = 0 = f(0, 0),$$

i.e., f is continuous in $\langle 0, 0 \rangle$. □

Problem 8 **(6p)**

Let $\mathbf{v} : \mathbb{R}^2 \setminus \{\langle 0,0\rangle\} \to \mathbb{R}^2$ be the vector field

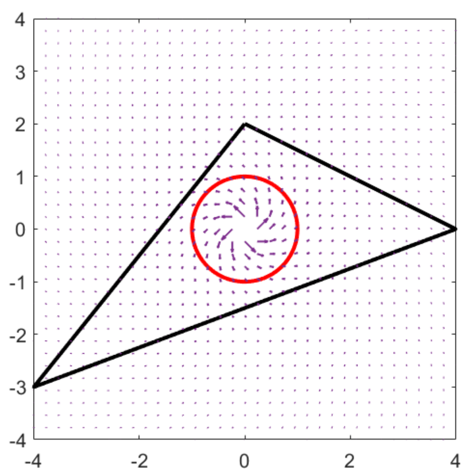
$$\mathbf{v}(x,y) = \begin{bmatrix} \frac{x-y}{x^2+y^2} \\ \frac{y+x}{x^2+y^2} \end{bmatrix}.$$

(a) Calculate the value of (1p)

$$\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}$$

(b) Calculate the line integral of $\mathbf{v}$ along the unit circle, traversed counter-clockwise. (2p)

(c) Now consider a triangle T with two corners corners $\langle 0,2\rangle$, $\langle 4,0\rangle$ and a third corner $\langle a,b\rangle$ which is chosen so that the triangle contains the unit circle, as in the figure. Calculate the line integral of $\mathbf{v}$ along the border of T, again traversed counter-clockwise. (3p)



Tip Neither of the line integrals are equal to zero.

Solution. (a) Let's simply do the calculation

$$\begin{aligned}\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} &= -\frac{1}{x^2+y^2} - \frac{2y(x-y)}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2x(x+y)}{(x^2+y^2)^2} \\ &= \frac{-2(x^2+y^2) + 2y^2 + 2x^2 - 2xy + 2xy}{(x^2+y^2)^2} = 0.\end{aligned}$$

(b) The fact that $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$ would suggest that an application of Green's theorem to the unit disc would yield that the line integral is zero. However, Green is not applicable here, since $\mathbf{v}$ is not defined in the origin.

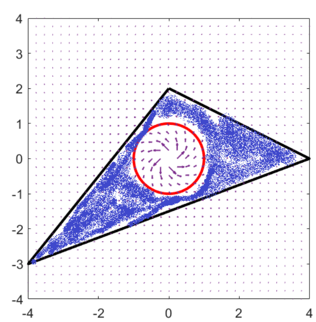
We must instead calculate the integral by hand. A parametrization of the unit circle, with the correct orientation, is given by $\gamma(\theta) = \langle \cos(\theta), \sin(\theta) \rangle, 0 \leq \theta \leq 2\pi$. Its derivative is given by $\gamma'(\theta) = \langle -\sin(\theta), \cos(\theta) \rangle$, and $\gamma(\theta) = \left\langle \frac{\cos(\theta)-\sin(\theta)}{\cos^2(\theta)+\sin^2(\theta)}, \frac{\sin(\theta)+\cos(\theta)}{\cos^2(\theta)+\sin^2(\theta)} \right\rangle = \langle \cos(\theta) - \sin(\theta), \sin(\theta) + \cos(\theta) \rangle$. Thus

$$\begin{aligned}
\int_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \langle \cos(\theta) - \sin(\theta), \sin(\theta) + \cos(\theta) \rangle \cdot \langle -\sin(\theta), \cos(\theta) \rangle \, \mathrm{d}\theta \\
&= \int_0^{2\pi} -\cos(\theta)\sin(\theta) + \sin^2(\theta) + \cos(\theta)\sin(\theta) + \cos^2(\theta) \, \mathrm{d}\theta = \int_0^{2\pi} 1 \, \mathrm{d}\theta = 2\pi.
\end{aligned}$$

(c) Let S be the area between the triangle and the circle, i.e. the blue area in the sketch below. In contrast to above, Greens theorem can be applied on S, since $\mathbf{v}$ is defined and differentiable on its entirety. It says that

$$\int_{\partial S} \mathbf{v} \cdot d\mathbf{r} = \int_S \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} r \right) \mathrm{d}x\mathrm{d}y = 0,$$

where we in the final step applied (a).



The boundary of S consists of the union of the unit circle and T. In order for Greens theorem to be true, we thereby need to traverse T in a counter-clockwise fashion, and the unit circle in a clockwise fashion. Since we above calculated the line integral for the opposite direction, we get

$$0 = \int_T \mathbf{v} \cdot d\mathbf{r} - \int_\gamma \mathbf{v} \cdot d\mathbf{r} = \int_T \mathbf{v} \cdot d\mathbf{r} - 2\pi.$$

Thus

$$\int_T \mathbf{v} \cdot d\mathbf{r} = 2\pi.$$

Remark: Another route is to notice that the two curves are homotopic, and that (a) shows that the curve obeys the integrability criterion. This immediately implies that the two line integrals must be equal. $\square$