Examples of Problem Solving Exam Exercises LMA017

In all exercises, it is of high importance to justify your answers! Just giving the correct answer will normally result in zero points.

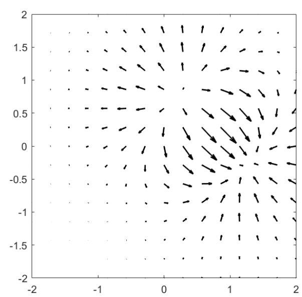
Most of the exercises are in the 'red zone' when it comes to difficulty that is acceptable on an exam. Some of them are well above it – 4,13 and 14 are definitely in this category. Less than a third of the exam will be problems of this type! More exact info can be retrieved from the exercise exam, which will be posted later.

No guarantee of correctness!

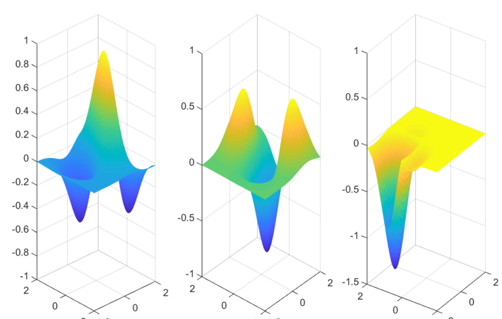
Week 1

1. Functions of several variables, Visualization

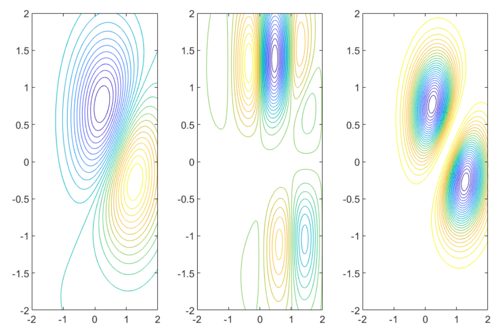
The following plot depicts the gradient ∇f of a function $f : \mathbb{R}^2 \to \mathbb{R}$.



(a) Which one of following graphs shows $\frac{\partial f}{\partial x}$?



(b) Which one of the following contour plots depicts f?



Solution. (a) $\frac{\partial f}{\partial x}$ is the first component of $\nabla f(x, y)$. Looking at the quiver plot, we see that the x-components of the arrow are negative in the left of the domain, then tend to be positive, and then negative again. The only plot that behaves like this is the **leftmost** plot, whence that must be the depiction of $\frac{\partial f}{\partial x}$.

(b) The gradient shows in which direction the function is growing the most. Looking at the quiver plot, we hence must conclude that the function has a minimum in the upper left of the domain, and a maximum in the lower right of the domain. It is only the **leftmost plot** that shows a function of this form - notice that the middle one has more than one critical point, and the right one has to maxima or minima, since the level lines near the two critical points have the same color. □

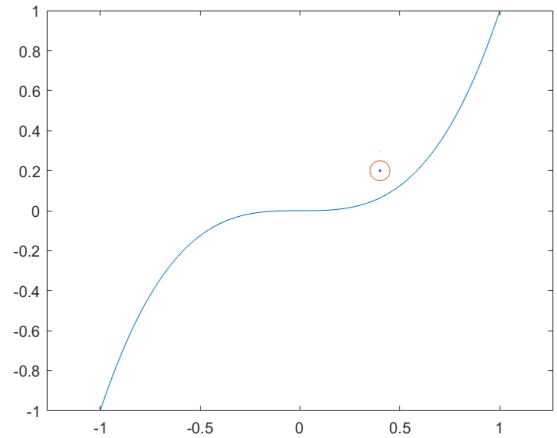
2. Limits and continuity

Consider the following function

$$f(x, y) = \begin{cases} 0 & \text{if } y = x^3 \\ 1 & \text{else.} \end{cases}$$

Determine the points in which f is continuous.

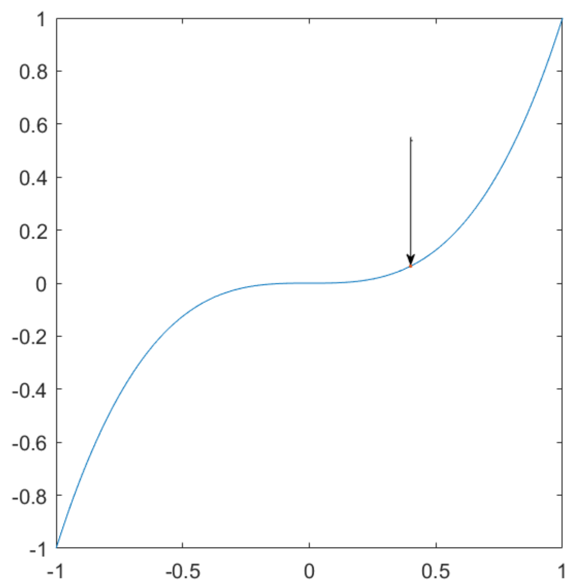
Solution. First, let $< x_0, y_0 >$ be a point for which $y_0 \neq x_0^3$. Then there exists a small neighbourhood U around $< x_0, y_0 >$ so that $y \neq x^3$ for all $< x, y > \in U$. Thus, $f(x, y) = 1 = f(x_0, y_0$ for all points in the neighbourhood. Consequently, for all $\epsilon > 0$, we have $\epsilon f(x, y) - f(x_0, y_0) = 0 < \epsilon$ for $< x, y >$ in the neighbourhood U. This is the definition of f being continuous in $< x_0, y_0 >$.



Now let $< x_0, y_0 >$ be a point in which $y_0 = x_0^3$, so that $f(x_0, y_0) = 0$. For any $h > 0$, we will then have $y_0 + h \neq x_0^3$, so that $f(x_0, y_0 + h) = 1$. Consequently

$$\lim_{h \to 0^+} f(x_0, y_0 + h) = \lim_{h \to 0^+} 1 = 1 \neq 0 = f(x_0, y_0).$$

Thus, if we approach the point $< x_0, y_0 >$ from above, $f(x, y)$ does not tend to $f(x_0, y_0)$. This proves that f is *not continuous* in such a point.



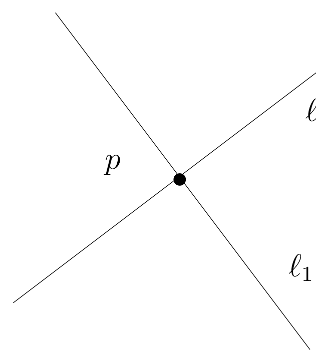
Thus, the function is continuous in the points where $y \neq x^3$, and else not.

$\square$

Week 2

3. Derivatives I

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has the following property: There exists two non-parallell lines ℓ_1 and ℓ_2 on which f is constantly equal to 1. Let $\mathbf{p}$ be the intersection of the two lines.



(a) Let $\mathbf{v}_1$, $\mathbf{v}_2$ be the directional vectors of the lines. Show that $D_{\mathbf{v}_1} f(\mathbf{p}) = D_{\mathbf{v}_2} f(\mathbf{p}) = 0$.

(b) Assume that f is differentiable. Show that $\nabla f(\mathbf{p}) = \mathbf{0}$.

(c) Does f have to be differentiable? Show this, or give a counterexample. If you give a counterexample, make sure to specify the lines ℓ_1 and ℓ_2.

Solution. (a) Due to the fact that f is equal to one on the lines, we have $f(\mathbf{p} + t\mathbf{v}_i) = 1$ for all $t \in \mathbb{R}$ and all $i = 1, \ldots, 2$. The definition of the directional derivative therefore implies

$$\lim_{t \to 0} \frac{f(\mathbf{p} + t\mathbf{v}_i) - f(\mathbf{p})}{t} = \lim_{t \to 0} \frac{1 - 1}{t} = \lim_{t \to 0} \frac{0}{t} = \lim_{t \to 0} 0 = 0.$$

(b) We have the following relation between the gradient $\nabla f(\mathbf{p})$ and the directional derivatives $D_{\mathbf{v}_i} f(\mathbf{p})$:

$$D_{\mathbf{v}_i} f(\mathbf{p}) = \mathbf{v}_i \cdot \nabla f(\mathbf{p}) = \mathbf{v}_i^T \nabla f(\mathbf{p})$$

Thus, if we define a matrix A whose rows are equal to $\mathbf{v}_1^T$ and $\mathbf{v}_2^T$, we have

$$A \nabla f(\mathbf{p}) = \mathbf{0}.$$

Due to the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallell, the matrix A is invertible. Thus, the above implies that

$$\nabla f(\mathbf{p}) = A^{-1}\mathbf{0} = \mathbf{0}.$$

$\square$

4. Derivatives II

Two hiking groups are moving through a landscape. The height of the landscape is described by a differentiable function h. The trajectories of the two groups are described by

$$\gamma_1(t) =< t, t^2 >, \text{ and } \quad \gamma_2 =< t, 3t >, \text{ respectively.}$$

Time is measured in minutes. The two groups record how their heights are changing with time. At the time $t = 0$ min, when the two groups meet in the point $< 0, 0, h(0, 0) >$, group 1 reports that their height is rising with a rate 0.1 m/min and group 2 reports that their height is rising with 0.4 m/min.

(a) Use the chain rule to write down a formula involving $h(0, 0)$, $\nabla h(0, 0)$, $\gamma_1(0)$, and $\gamma_1'(0)$ that expresses the rate of height change the first hiking group is experiencing at $t = 0$.

(b) What is the gradient of h in $(0, 0)$?

Solution. (a) The height that the first hiking group is at at time t is given by $h \circ \gamma_1(t)$. The chain rule implies

$$(h \circ \gamma_1)'(t) = h'(\gamma_1(t))\gamma_1'(t) = \nabla h(\gamma_1(t)) \cdot \gamma_1'(t).$$

Plugging in $t = 0$, we get

$$(h \circ \gamma_1)'(0) = \nabla h(\gamma_1(0)) \cdot \gamma_1'(0) = \nabla h(0, 0) \cdot \gamma_1'(0).$$

(b) By the exact same argument as above, the rate of change the second hiking group is given by $\nabla h(0, 0) \cdot \gamma_2'(0)$. Hence,

$$\begin{aligned} 0.1 &= \nabla h(0, 0) \cdot \gamma_1'(0) \\ 0.4 &= \nabla h(0, 0) \cdot \gamma_2'(0) \end{aligned}$$

Let us calculate the derivatives of the two γ-maps:

$$\begin{aligned} \gamma_1'(t) &= \begin{bmatrix} 1 \\ 2t \end{bmatrix} \implies \gamma_1'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \gamma_2'(t) &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \implies \gamma_1'(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Consequently, if we write $\nabla h(0, 0) =< a, b >$:

$$\begin{aligned} 0.1 &= \nabla h(0, 0) \cdot \gamma_1'(0) = 1 \cdot a + 0 \cdot b \\ 0.4 &= \nabla h(0, 0) \cdot \gamma_2'(0) = 1 \cdot a + 3 \cdot b. \end{aligned}$$

This linear system of equations is easily solved – its unique solution is $a = b = 0.1$. Hence,

$$\nabla h(0, 0) =< 0.1, 0.1 > .$$

□

5. The Schrankensatz

Two drones are moving through the air. Their positions relative to a base station are determined by a gps. They are both equipped with altimeters, which determine their heights above sea level. At a certain time, the measurement reads as follows

- Drone 1: Position $< 200, 400 >$, height 35.
- Drone 2: Position $< -80, 130 >$, height 50.

All readings are in meters. We can assume that the errors of the gps measurements of each coordinate is not larger than 5 m, and that the height measurements are off by at most 1 meter.

Use the Schrankensatz to determine the (three-dimensional) distance between the drones, with error bounds.

Solution. If the positions of the drones are denoted by $< a_1, a_2, a_3 >$ and $< b_1, b_2, b_3 >$, the distance between them is given by

$$D(a_1, a_2, a_3, b_1, b_2, b_3) = |< a_1, a_2, a_3 > - < b_1, b_2, b_3 >| = |< a_1 - b_1, a_2 - b_2, a_3 - b_3 >|$$
$$= \sqrt{(a_1 - b_2)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}.$$

If we denote the measured values with bars, $\bar{c}$, and the actual values with tildes, $\tilde{c}$, we have

$$\bar{a}_1 = 200, \tilde{a}_2 \in [195, 205], \quad \bar{a}_2 = 400, \tilde{a}_2 \in [395, 405], \quad \bar{a}_3 = 35, \tilde{a}_3 \in [34, 36].$$
$$\bar{b}_1 = -80, \tilde{b}_2 \in [-85, -75], \quad \bar{b}_2 = 130, \tilde{b}_2 \in [125, 135], \quad \bar{b}_3 = 50, \tilde{b}_3 \in [49, 51].$$

The Schrankensatz implies that

$$\left| D(\bar{\mathbf{a}}, \bar{\mathbf{b}}) - D(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \right| \leq \sum_{i=1}^{3} M_{a_i} \left| \bar{a}_i - \tilde{a}_i \right| + \sum_{i=1}^{3} M_{b_i} \left| \bar{b}_i - \tilde{b}_i \right|,$$

Where the M_{a_i} and M_{b_i}-parameters are upper bounds for the partial derivatives over all possible values. We calculate the partial derivatives:

$$\frac{\partial D}{\partial a_i} = \frac{2(a_i - b_i)}{2\sqrt{(a_1 - b_2)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}}$$
$$\frac{\partial D}{\partial a_i} = \frac{2(b_i - a_i)}{2\sqrt{(a_1 - b_2)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}}$$

Now we need to find bounds for the absolute values of these entities. We do this as follows: for the numerators, we choose a_i and b_i as far away from each other as they can be, e.g. $a_1 = 205, b_1 = -85$. In the denominator, we instead need to choose them as close to each other as possible, e.g. $a_1 = 195$ and $b = -75$ (in order to make the denominator as small as possible). We get

$$
\begin{aligned}
M_{a_1}, M_{b_1} &\le \frac{|205 - (-85)|}{\sqrt{(195-(-75))^2 + (395-135)^2 + (36-49)^2}} \approx 0.7732 \\
M_{a_2}, M_{b_2} &\le \frac{|405 - (125)|}{\sqrt{(195-(-75))^2 + (395-135)^2 + (36-49)^2}} \approx 0.7465 \\
M_{a_3}, M_{b_3} &\le \frac{|34 - 51|}{\sqrt{(195-(-75))^2 + (395-135)^2 + (36-49)^2}} \approx 0.0453
\end{aligned}
$$

Using $|\overline{a}_1 - \tilde{a}_1|, |\overline{a}_2 - \tilde{a}_2|, \left|\overline{b}_1 - \tilde{b}_1\right|, \left|\overline{b}_2 - \tilde{b}_2\right| \le 5$ and $|\overline{a}_3 - \tilde{a}_3|, \left|\overline{b}_3 - \tilde{a}_3\right| \le 1$, we thus obtain

$$\sum_{i=1}^{3} M_{a_i} |\overline{a}_i - \tilde{a}_i| + \sum_{i=1}^{3} M_{b_i} \left|\overline{b}_i - \tilde{b}_i\right| \le 15.29.$$

Since the measured value $D(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \approx 382.13$, we obtain

$$366.84 \le D(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}) \le 397.43.$$

□

Week 3

6. Optimization I

Consider the function

$$g : \mathbb{R}^2 \to \mathbb{R}, < x, y > \mapsto (x^2 + y^2) e^{-x^2 - y^2}$$

(a) Determine and classify all stationary points of g.

(b) Does g have a global maximum? A global minimum?

Solution. (a) Let us simplify the calculation slightly by using the chain rule to our advantage. If we define $f(t) = te^{-t}$ and $h(x, y) = x^2 + y^2$, we have $g = f \circ h$, and consequently

$$g'(x, y) = f'(h(x, y)) h'(x, y) \quad \Rightarrow \quad \nabla g(x, y) = \nabla h(x, y) f'(h(x, y))$$

We will need also the second derivative to classify the stationary points. We therefore apply the chain rule and the product rule to get

$$
\begin{aligned}
g''(x, y) &= (\nabla h(x, y))' f'(h(x, y)) + \nabla h(x, y) (f' \circ h)'(x, y) \\
&= h''(x, y) f'(h(x, y)) + \nabla h(x, y) f''(h(x, y)) h'(x, y).
\end{aligned}
$$

In order to apply these formulas, we need the first and second derivatives of f and h. We have

$$f'(t) = e^{-t} - te^{-t}, f''(t) = -2e^{-t} + te^{-t}$$
$$h'(x,y) = \begin{bmatrix} 2x & 2y \end{bmatrix}, \nabla h(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}, h''(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Now let us find the critical points, i.e. the points for which $\nabla g = \mathbf{0}$. According to the above formula, we have $\nabla g(x,y) = f'(h(x,y))\nabla h(x,y)$. Thus, in order for g to vanish, we need either $f'(h(x,y)) = 0$ or $\nabla h(x,y) = \mathbf{0}$. The latter only happens in $\langle x, y\rangle = \langle 0, 0\rangle$. The former occurs when

$$0 = f'(h(x,y)) = e^{-h(x,y)} - h(x,y)e^{-h(x,y)} = e^{-(x^2+y^2)}(1 - x^2 - y^2),$$

i.e., in points where $x^2 + y^2 = 1$, the unit circle. The critical points of the function are thus given by the origin and all points on the unit circle.

In order to classify the points, we know evaluate the second derivative in the points. Let us start with the origin. We have $f'(h(0,0)) = f'(0) = e^{-0} - 0 \cdot e^{-0} = 1$ and $\nabla h(0,0) = \langle 0, 0\rangle$ so that

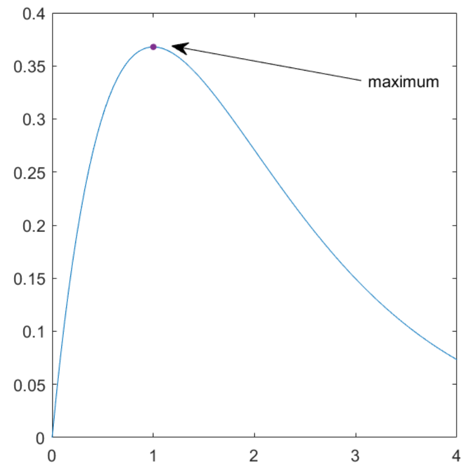
$$g''(0,0) = h''(0,0)f'(h(0,0)) + \nabla h(0,0)f''(h(0,0))h'(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

This matrix has a positive determinant, and the pure second derivative with respect to x is $2 > 0$. The matrix is thus positive definite, which implies that the point is a local minimum.

We move on to the point on the unit circle. We have $h(x,y) = x^2 + y^2 = 1$ for such points, so that $f'(h(x,y)) = f'(1) = e^{-1} - e^{-1} = 0$, and $f''(h(x,y)) = f''(1) = -2e^{-1} + 1 \cdot e^{-1} = -e^{-1}$. Consequently

$$\begin{aligned} g''(x,y) &= h''(x,y)f'(1) + \nabla h(x,y)f''(1)h'(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot 0 - 2e^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix} \\ &= -2e^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix} = -2e^{-1}\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \end{aligned}$$

This matrix has a zero determinant – this can either be seen through calculation, or the fact that it has a one-dimensional range (so that it is not injective). Thus, our endevours have been useless – we instead need to do something else.



However, the fact that $g = f \circ h$ actually makes us realise that the points in the unit circle are local maxima. Looking at the graph of the function f, we see that it has a (global) maximum in $t = 1$. Therefore, for any $< \tilde{x}, \tilde{y} >$ (close to $\langle x, y \rangle$), we have

$$f(h(\tilde{x}, \tilde{y})) = f(\tilde{x}^2 + \tilde{y}^2) \leq f(1) = f(x, y),$$

which is the definition of a maximum.

(b) f has a global maximum in $t = 1$ – looking at f', it is easily seen that f is decreasing on $[1, \infty[$ and increasing on $[0, 1]$. Consequently, $g(h(x, y)) \leq f(1)$, and $f(1)$ is attained on the unit circle. Thus, g has the global maximum value e^{-1}, attained in all points on the unit circle. Furthermore, the local minimum on the origin is in fact a global minimum: g is clearly nonnegative, so that

$$g(x, y) \geq 0 = g(0, 0)$$

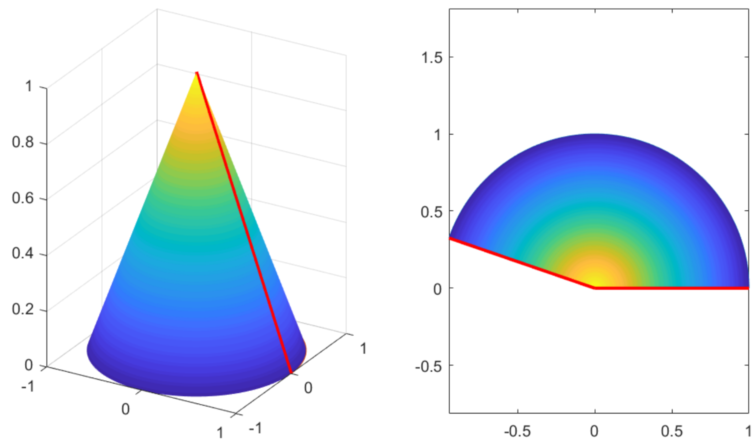
for all $\langle x, y \rangle \in \mathbb{R}^2$. □

7. Optimization II

Assume that we are given a fixed amount of batter to make an ice-cream cone. How should we design the cone to fit as much icecream as possible in the cone?

Solution. The fixed amount of batter means that the surface area of the cone is constrained to be equal to a constant value, say C, which is positive. Our task is to optimize the volume under this constraint.

There are several ways to choose parameters which determines the cone. One way that is especially appealing to this task is to use the length of the length of the *generatrix* r (the red line in the figure below). If we 'roll out' the surface of the cone, we then obtain a circle sector (since the all the points on the circle encompassing the base of the cone have the same distance to the apex). The cone is then determined by the size of the angle of that circle. We will use the fraction of the angle to 2π, $\alpha \in [0,1]$, as the second parameter describing the cone.



In these parameters, the surface area is given by $\pi r^2 \alpha$ (we have a fraction α of a whole disc of radius r). The volume of the cone is given by $\frac{\pi}{3} h \rho^2$, where ρ is the radius of the base of the cone. We need to express ρ and h in r and α. Since the circumference of the base is both given by $2\pi\rho$ and $\alpha \cdot 2\pi r$ (it's a fraction alpha of the circumference of a circle of radius r, we have $\rho = \alpha r$. The pythagorean theorem furthermore implies that

$$h^2 + \rho^2 = r^2 \Rightarrow h = \sqrt{r^2 - \rho^2} = r\sqrt{1-\alpha^2}.$$

Thus, the optimization problem we in the end need to solve is

$$\max \frac{\pi}{3} r\sqrt{1-\alpha^2}\alpha^2 r^2 \text{ subject to } \pi r^2 \alpha = C$$

To simplify the calculations, let's remove the constants π and $\pi/3$ - the former can be accomplished by redefining C, and the latter by simply noting that the above expression is maximize exactly when $r\sqrt{1-\alpha^2}\alpha^2 r^2$ - it's just a multiplicative constant. Thus, we concentrate on

$$\max \alpha^2\sqrt{1-\alpha^2} r^3 \text{ subject to } r^2\alpha - C = 0, 0\alpha \leq 1, r \geq 0$$

We use the conventional notation $f(r,\alpha) = \alpha^2\sqrt{1-\alpha^2}r^3$ and $g(r,\alpha) = r^2\alpha - C$.

Let us first convince that this problem has a solution. Let us first note that the constraint defines a curve in the plain, which clearly is a closed set. It is however not a bounded set - r can tend to ∞ as long as α tends to 0 at the same time. We hence cannot use the standard compactness argument. To fix this, let's control what happens when $r \to \infty$. Since $\alpha = C/r^2$ due to the constraint, we thus need to check what happens to

$$\frac{C^2\sqrt{1-\frac{C^2}{r^4}}r^3}{r^4} = \frac{\sqrt{1-\frac{C^2}{r^4}}}{r}.$$

That function however clearly tends to 0 as $r \to \infty$. Thus, if $R \geq R_0$, f is smaller than the value in, say, $(1, 1/C^2)$, and thus smaller than the largest value on the compact set $r^2\alpha - C = 0, 0 \leq \alpha \leq 1, R_0 \geq r \geq 0$, which exists – f is continuous.

Now that we know that the function has a largest value, we may use the method of Lagrange multipliers to find it. Here we formally need to take extra care of the point boundary case $\alpha = 1$, but the formula for f shows that the maximum clearly does not occur there – $f(r, 1) = 0$. We may thus concentrate on the Lagrange criterion $\nabla f = \lambda \nabla g$, which reads

$$3r^2\alpha^2\sqrt{1-\alpha^2} = 2\lambda\alpha r$$

$$\left(2\alpha\sqrt{1-\alpha^2} - \frac{\alpha^3}{\sqrt{1-\alpha^2}}\right) r^3 = \lambda r^2$$

The constraint $r^2\alpha = C > 0$prohibits α and r from being zero - which also shows that the gradient of the g function does not vanish on the set of feasible points. We may thus simply divide by them in both equations to obtain

$$\begin{aligned} 2\lambda = 2\left(2\alpha\sqrt{1-\alpha^2} - \frac{\alpha^3}{\sqrt{1-\alpha^2}}\right) r &= 3r\alpha\sqrt{1-\alpha^2} \\ \iff 2\left(2 - \frac{\alpha^2}{1-\alpha^2}\right) &= 3 \\ \iff 4(1-\alpha^2) - 2\alpha^2 &= 3(1-\alpha^2) \\ \alpha^2 &= \frac{1}{3}. \end{aligned}$$

Since the function gets smaller as we send α to either 0 or 1, we conclude that this point must be a maximum. The optimal design is thus given by $\alpha = \frac{1}{\sqrt{3}}$. □

8. Optimization III

Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function.

(a) Show that g has a maximum on the unit sphere.

(b) Show that there must exist a point $\mathbf{p}$ on the sphere in which $\nabla g(\mathbf{p})$ is parallel to $\mathbf{p}$.

Solution. (a) The unit sphere is a closed set - its boundary is equal to itself. It is clearly also bounded. Thus, the unit sphere is compact. Since furthermore g is differentiable, it is also continuous. The existence of a maximum thus follows from the theorem of maximum and minimum values on compact sets.

(b) The maximum on the unit sphere is a solution of the constrained optimization problem

$$\max g(x, y, z) \text{ subject to } x^2 + y^2 + z^2 - =: h(x, y, z) = 0.$$

The gradient of the function h is $\nabla h(x, y, z) = 2 < x, y, z >$, i.e. $\nabla h(\mathbf{p}) = 2\mathbf{p}$. It thus never vanishes on the unit sphere, and we may apply the Lagrange theorem, which states that in the point $\mathbf{p}_0$ where the maximum occurs, we must have

$$\nabla g(\mathbf{p}_0) = \lambda \nabla h(\mathbf{p}_0) = 2\lambda \mathbf{p}_0,$$

which is exactly is what to be shown - the gradient of g at $\mathbf{p}_0$ is parallell to $\mathbf{p}_0$. □

9. Optimization IV

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Suppose that both functions

$$g(t) = f(t, 0) \text{ and } h(t) = f(0, t)$$

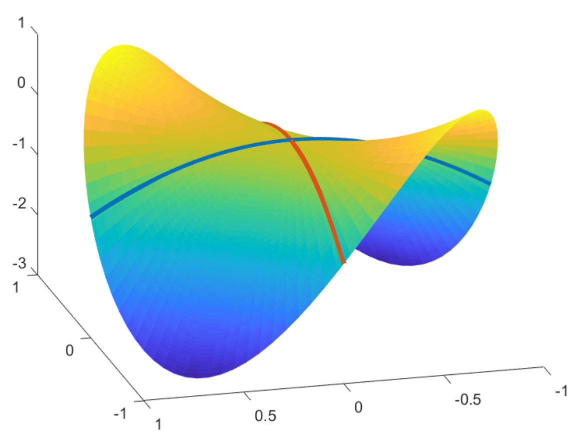
have maximums in $t = 0$. Does f need to have a maximum in $(0, 0)$? Prove it or give a counterexample.

Solution. The claim is false. To see this, consider the example $f(x, y) = xy$. This function does not have a maximum in $\langle 0, 0\rangle$ – it is easily shown via the second derivative test that the origin is a saddle point. The two functions g and h are however the zero functions, which has a maximum in $t = 0$.

It is in fact also possible to construct an example where g and h have strict local maxima in 0, for instance $f(x, y) = -x^2 + -y^2 + 4xy$. The origin is again a critical point, but since the Hessian of f

$$\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$$

has a negative determinant, it is again a saddle and not a maximum. The two functions $g(t) = h(t) = -t^2$ however have strict minima in $t = 0$.



□

Week 4

10. Integrals and probability

We choose a point $\mathbf{p} =< x, y, z >$ uniformly at random in the unit cube $[0,1]^3$. What is the probability that z is larger than $x^2 + y^2$?

Solution. Since the point is chosen *uniformly* at random, the vector has a probability density $\rho(x,y,z) = 1$. If we define the set $E = \{\langle x, y, z\rangle \mid z \geq x^2 + y^2\}$, we therefore have

$$\mathbb{P}\left(z \geq x^2 + y^2\right) = \mathbb{P}\left(\langle x, y, z\rangle \in E\right) = \iiint_E 1 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

We now apply the theorem of Fubini - in the set E, the x values range from 0 to 1. For each x, y ranges from 0 to 1, and for each pair $\langle x, y\rangle$, z ranges from $x^2 + y^2$ to 1. Therefore

$$\iiint_E 1 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^1 \int_{x^2+y^2}^1 \mathrm{d}z\mathrm{d}y\mathrm{d}x$$

$$= \int_0^1 \int_0^1 (1 - x^2 - y^2)\mathrm{d}y\mathrm{d}x = \int_0^1 \left[(1-x^2)y - \frac{y^3}{3}\right]_{y=0}^{y=1} \mathrm{d}x$$

$$= \int_0^1 1 - x^2 - \frac{1}{3} \,\mathrm{d}x = \left[\frac{2}{3}x - \frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{1}{3}.$$

□

11. Integrals

The unit circle is filled with a material whose density is varying according to

$$\rho(x,y) = \frac{1}{1 + 0.8 \cdot (x^2 + y^2) - 0.1 \cdot \cos(x)y + 0.1 \cdot y\cos(y)}$$

Show that the circle weighs more than

$$\frac{2\pi}{0.8} \cdot \ln(2) \text{ mass units..}$$

Solution. **This problem was not properly designed** The idea was to argue that since $y\cos(x)$ and $y\sin(y)$ both are smaller than 1 in modulus on the unit disc, we have

$$\begin{aligned}\rho(x,y) &= \frac{1}{1 + 0.8 \cdot (x^2 + y^2) - 0.1 \cdot \cos(x)y + 0.1 \cdot y\cos(y)} \geq \frac{1}{1 + 0.8 \cdot (x^2 + y^2) - 0.1 - 0.1} \\ &= \frac{1}{0.8(1 - x^2 - y^2)}.\end{aligned}$$

Consequently, if we denote the unit disc as C,

$$\iint_C \rho(x,y)\,\mathrm{d}x\mathrm{d}y \geq \iint_C \frac{1}{0.8(1-x^2-y^2)}\,\mathrm{d}x\mathrm{d}y$$

In order to solve the final integral, we change to polar coordinates

$$\iint_C \frac{1}{0.8(1-x^2-y^2)}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{2\pi} \frac{r}{0.8(1-r^2)}\,\mathrm{d}\theta\mathrm{d}r = \frac{2\pi}{0.8}\int_0^1 \frac{r}{1-r^2}dr = \frac{2\pi}{0.8}\left[-\frac{1}{2}\ln(1-r^2)\right]_{r=0}$$

The final limit does not exist, so the integral does not exist. Hence, **this task was flawed**. □

12. Variable substitution I

Determine the area of the domain in the plane which is bounded by the curves $y = 2/x$, $y = 1/x$, $y^2 = x^2 + 1$ and $y^2 = x^2 + 2$. *Tip:* Use variable substitution!

Solution. Let ϕ be the map

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2\ \langle x, y\rangle \mapsto \left\langle xy, y^2 - x^2\right\rangle$$

If we denote the domain whose area we are supposed to determine with D, and $\phi(x,y) =< s, t >$, we see that

$$\langle x, y\rangle \in D \ \Leftrightarrow\ \phi(x,y) =< xy, y^2 - x^2 >\in [1,2]\times[1,2]$$

In other words, $\phi(D) = [1,2]\times[1,2]$. Thus, for any function f

$$\iint_{\phi(D)} f(s,t)\,\mathrm{d}s\mathrm{d}t = \iint_D f(\phi(x,y))\det(\phi'(x,y))\,\mathrm{d}x\mathrm{d}y$$

Let us calculate the functional determinant of ϕ:

$$\det(\phi'(x,y)) = \begin{pmatrix} y & x \\ -x & y \end{pmatrix} = y^2 + x^2$$

Thus

$$\iint_{\phi(D)} f(s,t)\,\mathrm{d}s\mathrm{d}t = \iint_D f(\phi(x,y))(x^2+y^2)\,\mathrm{d}x\mathrm{d}y$$

Now, what we actually want to calculate is the integral of 1 over D. We thus need to find a function f so that

$$f(s,t) = f(\phi(x,y)) = \frac{1}{x^2+y^2}$$

What we need to do is to express x^2+y^2 in terms of s and t. We have

$$t^2 = (y^2-x^2)^2 = y^4 - 2x^2y^2 + x^4 = (y^2+x^2)^2 - 4(xy)^2 = (y^2+x^2)^2 - 4s^2,$$

so that

$$(x^2+y^2) = \sqrt{4s^2+t^2}.$$

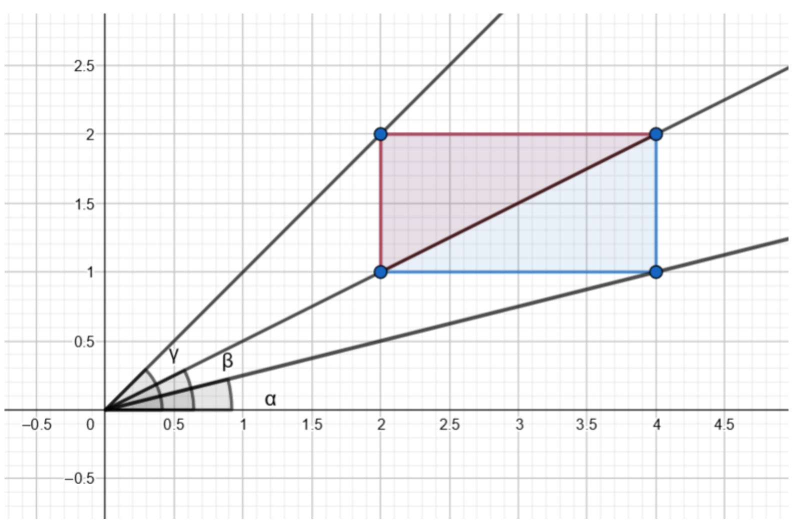
Thus, if we choose $f(s,t) = (4s^2+t^2)^{-.5}$, we have $f(s,t) = f(xy, y^2-x^2) = \frac{1}{x^2+y^2}$. Thus

$$\begin{aligned}\iint_D \frac{1}{x^2+y^2}(x^2+y^2)\,\mathrm{d}x\mathrm{d}y &= \iint_{\phi(D)} \frac{1}{\sqrt{4s^2+t^2}}\,\mathrm{d}s\mathrm{d}t \\ &= \int_1^2\int_1^2 \frac{1}{\sqrt{4s^2+t^2}}\,\mathrm{d}s\mathrm{d}t = \int_1^2\int_2^4 \frac{1}{2\sqrt{\tilde{s}^2+t^2}}\,\mathrm{d}\tilde{s}\mathrm{d}t\end{aligned}$$

In the final step, we made the substiution $\tilde{s} = 2s$.

The integrand makes it natural to go over to polar coordinates to solve the final integral. We then need to determine limits for the angles and the radius. The figure below shows that the θ-values in the square $[2,4]\times[1,2]$ range from $\alpha = \arctan\left(\frac{1}{4}\right)$ and $\gamma = \arctan(1) = \frac{\pi}{4}$. For each value of θ, r goes between two values. The way to determine these limits are different for $\theta < \beta$ and $\theta \geq \beta$, where $\beta = \arctan\left(\frac{2}{4}\right)$. For $\theta < \beta$, the two limiting criteria are $y = r\sin(\theta) = 1$ and $x = r\cos(\theta) = 4$. For $\theta \geq \beta$, they are instead $x = r\cos(\theta) = 2$ and $y = r\sin(\theta) = 2$. Thus, the correct description of the rectangle in polar coordinates are

$$\alpha \leq \theta \leq \beta, \frac{1}{\sin(\theta)} \leq r \leq \frac{4}{\cos(\theta)} \text{ or } \beta \leq \theta \leq \alpha, \frac{2}{\cos(\theta)} \leq r \leq \frac{2}{\sin(\theta)}$$



Thus,

$$\int_1^2\int_2^4 \frac{1}{2\sqrt{\tilde{s}^2+t^2}}\,\mathrm{d}\tilde{s}\mathrm{d}t = \int_\alpha^\beta \int_{\frac{1}{\sin(\theta)}}^{\frac{4}{\cos(\theta)}} \frac{r}{2r}\,\mathrm{d}r\mathrm{d}\theta + \int_\beta^\gamma \int_{\frac{2}{\cos(\theta)}}^{\frac{2}{\sin(\theta)}} \frac{r}{2r}\,\mathrm{d}r\mathrm{d}\theta$$
$$= \frac{1}{2}\int_\alpha^\beta \left(\frac{4}{\cos(\theta)} - \frac{1}{\sin(\theta)}\right)\mathrm{d}\theta + \frac{1}{2}\int_\beta^\gamma \left(\frac{2}{\sin(\theta)} - \frac{2}{\cos(\theta)}\right)\mathrm{d}\theta$$

Notice that the Jacobian worked to our advantage, making the r-integration very simple. Now, we 'only' need to work out the θ-integrals. This is done by cleverly substituting: for instance

$$\int \frac{1}{\cos(\theta)}\,\mathrm{d}\theta = \int \frac{\cos(\theta)}{\cos^2(\theta)}\,\mathrm{d}\theta = \int \frac{\cos(\theta)}{1-\sin^2(\theta)}\,\mathrm{d}\theta = [v=\sin(\theta)] = \int \frac{1}{1-v^2}dv$$
$$\frac{1}{2}\int\int \frac{1}{1-v} + \frac{1}{1+v}dv = \frac{1}{2}\left(\ln(1+v) - \ln(1+v)\right) + C = \frac{1}{2}\ln\left(\frac{1+\sin(\theta)}{1-\sin(\theta)}\right) + C.$$

Working out also the sin-terms, and inserting the β and γ-values, we obtain that the area is given by

$$\ln\left(\frac{1+\sin(\beta)}{1-\sin(\beta)}\right) - \ln\left(\frac{1+\sin(\alpha)}{1-\sin(\alpha)}\right) + \frac{1}{4}\left(\left(\frac{1+\cos(\beta)}{1-\cos(\beta)}\right) - \ln\left(\frac{1+\cos(\alpha)}{1-\cos(\alpha)}\right)\right)$$
$$+\frac{1}{2}\left(\left(\frac{1-\cos(\gamma)}{1+\cos(\gamma)}\right) - \ln\left(\frac{1-\cos(\beta)}{1+\cos(\beta)}\right)\right) + \frac{1}{2}\left(\ln\left(\frac{1-\sin(\gamma)}{1+\cos(\gamma)}\right) - \ln\left(\frac{1-\sin(\beta)}{1+\cos(\beta)}\right)\right)$$

□

Week 5

13. Variable substitution II

A positive function which is continuous everywhere on the unit ball except the origin is called *integrable* if the limit

$$\lim_{\epsilon\to 0+} \iiint_{C_\epsilon} f\,\mathrm{d}V < \infty.$$

Here, $C_\epsilon = \{\mathbf{p}\,|\,\epsilon \le |\mathbf{p}| \le 1\}$.

For which α is the function

$$f(x,y,z) = \frac{1}{(x^2+y^2+z^2)\alpha}$$

integrable over the unit ball?

Solution. We use spherical coordinates to rewrite

$$\iiint_{C_\epsilon} f\, \mathrm{d}V = \int_\epsilon^1 \int_0^\pi \int_0^{2\pi} \frac{\rho^2 \sin(\varphi)}{(\rho^2)^\alpha}\, \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}\rho = 4\pi \int_\epsilon^1 \rho^{2-2\alpha} \mathrm{d}\rho.$$

Now let us consider two cases. If $2 - 2\alpha \neq -1$, we have

$$\int_\epsilon^1 \rho^{2-2\alpha}\, \mathrm{d}r = \left[\frac{\rho^{3-2\alpha}}{3-2\alpha}\right]_\epsilon^1 = \frac{1}{3-2\alpha} - \lim_{\epsilon \to 0+} \frac{\epsilon^{3-2\alpha}}{3-2\alpha}.$$

The above limit exists if and only if $3 - 2\alpha > 0$ – since only then $\epsilon^{3-2\alpha} \to 0$. Otherwise, it tends to ∞ (remember that the boundary case $3 - 2\alpha = 0$ is not considered here, since the calculations above is not correct when $2 - 2\alpha = -1$).

In the case that $2 - 2\alpha = -1$, the integral turns into

$$\int_\epsilon^1 \rho^{-1}\, \mathrm{d}\rho = [\ln(\rho)]_\epsilon^1 = -\ln(\epsilon),$$

which tends to ∞ as $\epsilon \to 0+$. Thus, also in this case, the limit does not exist.

We conclude that the function is integrable if and only if $3 - 2\alpha > 0$. □

14. Variable substitution III

Determine the integral of

$$f(x, y, z) = \cos(x + y + z)$$

over the unit ball $\{\mathbf{p} \mid |p| \leq 1\}$. *Tip:* Rotate the problem!

Solution. Let us introduce a new coordinate system in which the vector $\langle 1, 1, 1\rangle$ points in the same direction as the new z-axis, but has the same unit length as the original system. The vector $\langle 1, 1, 1\rangle$ will then have the representation $\sqrt{3}\mathbf{e}_{\tilde{z}}$ in the new coordinate system – it points in the $\tilde{z}$-direction, and has the length $\sqrt{1^2 + 1^2 + 1^2}$. We therefore have

$$f(x, y, z) = \cos(\langle 1, 1, 1\rangle \cdot \langle x, y, z\rangle) = \cos(\sqrt{3}\mathbf{e}_{\tilde{z}} \cdot \langle \tilde{x}, \tilde{y}, \tilde{z}\rangle) = \cos(\sqrt{3}\tilde{z}).$$

Introducing a new *euclidean* coordinate system with the same unit lengths as the original one will not change the volume element. A formal argument is given by the fact that the rotation transformation has the form $\phi(\mathbf{p}) = Q\mathbf{p}$ for an orthogonal Q. The functional determinant of that transformation is $|\det(Q)| = 1$, since that is a feature of orthogonal matrices.

Rotating also does not change the shape of the unit ball B. Hence, the integral we want to determine is equal to

$$\iiint_B \cos(\sqrt{3}z)\, \mathrm{d}V.$$

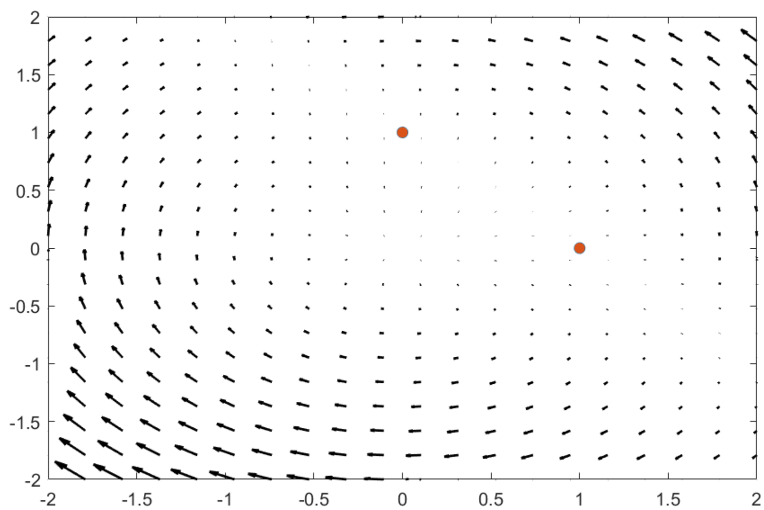
To solve this integral, we go over to spherical coordinates

$$\int_0^1 \int_0^\pi \int_0^{2\pi} \cos(\sqrt{3}\cos(\varphi))\rho^2 \sin(\varphi)\ \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}\rho = 2\pi \int_0^1 \int_0^\pi \cos(\sqrt{3}\cos(\varphi))\rho^2 \sin(\varphi)\ \mathrm{d}\varphi \mathrm{d}\rho$$
$$= 2\pi \int_0^1 \left[\frac{-\sin(\sqrt{3}\cos(\varphi))}{\sqrt{3}} \rho^2 \right]_{\varphi=0}^{\varphi=\pi} \mathrm{d}\rho = 2\pi \int_0^1 \frac{2\sin(\sqrt{3})}{\sqrt{3}} \rho^2\ \mathrm{d}\rho = \frac{4\pi \sin(\sqrt{3})}{3\sqrt{3}}.$$

□

15. Curve integrals

Below, a quiver plot of a vector field is shown. The two points $< 1, 0 >$ and $< 0, 1 >$ are marked. The vector field is zero on the line $x + y = 1$, and defined everywhere in $\mathbb{R}^2$.



(a) Describe a curve γ_2 between $< 1, 0 >$ and $< 0, 1 >$ for which

$$\int_{\gamma_1} \mathbf{v} \cdot d\mathbf{r} = 0.$$

(b) Sketch a curve γ_2 between $< 1, 0 >$ and $< 0, 1 >$ for which

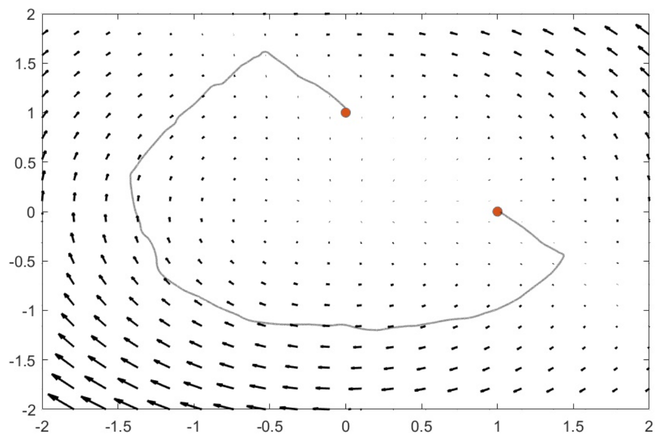
$$\int_{\gamma_2} \mathbf{v} \cdot d\mathbf{r} > 0.$$

(c) Can the vector field satisfy the differential equation

$$\frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial y}?$$

Solution. (a) We simply move along the straight line segment between $< 1, 0 >$ and $< 0, 1 >$. Since the vector field is zero along this line, $\mathbf{v}(\gamma(t)) \cdot \gamma'(t)$ will then always be zero, so that the line integral vanishes.

(b) See sketch. We move along a trajectory on which the derivative of γ points roughly in the same direction as $\mathbf{v}$ at all times. This will cause $\mathbf{v}(\gamma(t)) \cdot \gamma'(t)$ to be positive, which will make the line integral along the curve positive.



(c) No. Since the vectorfield is defined everywhere, if the differential equation would be satisfied, the field would be conservative (since the integrability criterion is sufficient on simply connected domains). Line integrals of conservative vector field are however path independent, which from (a) and (b) is clearly not the case. $\square$

Week 6 - 8

16. Flow integrals I

Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ a differentiable vector field. Assume that

$$\mathbf{v}(\mathbf{p}) \cdot \mathbf{p} < 0$$

for all $\mathbf{p} \in \mathbb{R}^3$. Can it be that $\operatorname{div}(\mathbf{v}) = 0$ in all of $\mathbb{R}^3$? Give an example of such a vector field, or prove that it cannot exist. *Tip*: Investigate the flow integral of $\mathbf{v}$ over the unit sphere.

Solution. The outward unit normal field of the unit sphere $\mathbb{S}^2$ has the form $\nu(\mathbf{p}) = \mathbf{p}$. This means that

$$\int_{\mathbb{S}^2} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \int_{\mathbb{S}^2} \mathbf{v}(\mathbf{p}) \cdot \mathbf{p} \, \mathrm{d}S < 0,$$

by the assumption of the problem. Now, if the divergence of the vector field is zero everywhere, Gauß's theorem would imply

$$\int_{\mathbb{S}^2} \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \int_B \mathrm{div}\mathbf{v}\ \mathrm{d}V = 0,$$

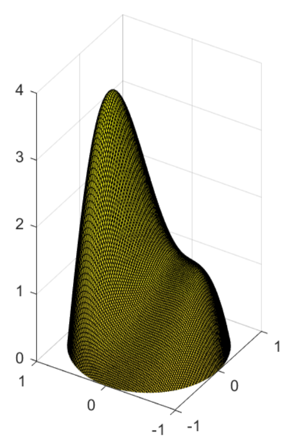
where B is the unit ball. This is a contradiction. Hence, the divergence cannot vanish everywhere.

$\square$

17. Flow integrals II

Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector fields

$$\mathbf{v}(x, y, z) = (x^2 + y^2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}(x, y, z) = \begin{bmatrix} -yx^2 \\ xy^2 \\ 0 \end{bmatrix}.$$



1. Show that $\mathbf{v} = \mathrm{curl}\mathbf{w}$.
2. Let K be the part of the graph of the function

$$g(x, y) = (1 - x^2 - y^2)e^{x+y}(1 + x^2)(1 + 4y^2)$$

which lies above the xy-plane (it is the surface shown in the figure). Calculate

$$\int_K \mathbf{v} \cdot d\mathbf{S}$$

Solution. (a) This can be solved through a straightforward calculation. A slight less calculation heavy route is to use the formula $\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v}$. We write

$$\mathbf{w} = xy \begin{bmatrix} -x \\ y \\ 0 \end{bmatrix}$$

The rotation of the vector field $< -x, y, 0 >$ is clearly zero (the x-component of the vector field does not depend on y or z, so both partial derivatives of it that are present in the curl formula vanish). Furthermore, $\nabla(xy) =< y, x, 0 >$. Thus

$$\nabla \mathbf{w} = \begin{bmatrix} y \\ x \\ 0 \end{bmatrix} \times \begin{bmatrix} -x \\ y \\ 0 \end{bmatrix} = (x^2 + y^2) \begin{bmatrix} 0\ 0 \\ 1 \end{bmatrix}$$

The last step is most easily seen via a straight calculation, which we in the interest of brevity not report here.

(b) We use the Stokes theorem

$$\iint_K \mathbf{v} \cdot d\mathbf{S} = \iint_K \mathrm{curl}\mathbf{w} \cdot d\mathbf{S} = \int_{\partial K} \mathbf{w} \cdot d\mathbf{s}.$$

Here, ∂K is the boundary curve of the surface, i.e. the unit circle in the xy-plane, which is parametrized by $\gamma(\theta) =< \cos(\theta), \sin(\theta), 0 >$, $\theta = 0 \leq 2\pi$. Since the orientation of K is not given by the task, we choose it so that it goes along with the application of the Stokes theorem. We now calculate

$$\int_{\partial K} \mathbf{w} \cdot d\mathbf{s} = \int_0^{2\pi} \begin{bmatrix} -\sin(\theta)\cos^2(\theta) \\ \cos(\theta)\sin^2(\theta) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} \mathrm{d}\theta = \int_0^{2\pi} 2\sin^2(\theta)\cos^2(\theta) d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} \sin^2(2\theta)\mathrm{d}\theta = \frac{1}{4}\int_0^{2\pi} 1 - \cos(4\theta)\ \mathrm{d}\theta = \frac{1}{8}.$$

□