Tentamen MVE255

Mathematical sciences, Chalmers University of Technology

Datum:	October 27th 2020, 14.00
Examiner:	Axel Flinth
Allowed aids:	Any.
Grade limits:	20 points for the grade 3. 30 points for the grade 4 40 points for the grade 5.

There are in total 50 points to collect.

Calculations and arguments should be presented in full. **Only providing an answer will normally not be rewarded with points.** Solutions may be written in Swedish or English.

If you use any external tool or resource, you should reference it. See the Canvas Page for more information. Not referencing properly may result in point deduction.

The exam consists of **eight (8)** problems. They are distributed over **four (4)** pages.

Good luck!

Problem 1 (6p)

Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be defined through

$$g(x, y, z) = \begin{bmatrix} x^2y + e^z \\ e^z xy \end{bmatrix}.$$

(a) Calculate the derivative of g. (4p)

(b) Evaluate g and g' in $\langle 1, 0, 1\rangle$. (2p)

Problem 2 (6p)

Calculate $\iiint_S z \,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where S is the part of the set

$$\{\langle x, y, z\rangle \mid 0 \leq x \leq y \leq 1, 0 \leq z \leq 1\}$$

which lies above the graph of $f(x, y) = xy$.

Problem 3 (6p)

Calculate $\iiint_K z\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where K is the cone $\{\langle x, y, z\rangle \mid x^2 + y^2 \leq z^2 \leq 1,\ z \geq 0\}$.

Problem 4 **(8p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined through and

$$f(x, y) = e^{-x^2} e^{y^2}$$

(a) Find and classify all critical points of the function on $\mathbb{R}^2$. (4p)

(b) Find the maximum and minimum of the function on the closed unit disc (4p)

$$D = \left\{ \langle x, y \rangle \mid x^2 + y^2 \leq 1 \right\}.$$

Problem 5 **(6p)**

If an object is thrown from ground level with an initial velocity v and an elevation angle α, the object will follow a parabola and land after (disregarding wind resistance etc.)

$$\ell = \frac{v^2 \sin(2\alpha)}{g} \text{ m.}$$

meters. $g = 9.8$ m/s^2 is the gravity factor.

A discus thrower is known to release the discus at a speed $v = 18$ m/s and an angle $\alpha = \pi/4$ on average. The speed can deviate up to 1 m/s, and the angle $\pi/12$ radians. Use the Schrankensatz and the above formula to give an estimate of where the discus will land, with error bounds.

Use the formula. Hence, ignore both wind resistance etc. and the fact that the hand of the discus thrower is above ground level at the time of release.

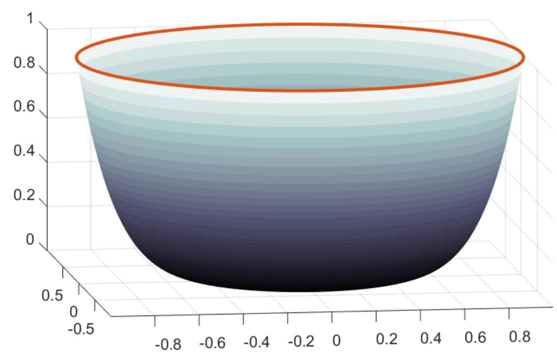
Problem 6 **(6p)**

Consider the vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{v}(x, y, z) = \begin{bmatrix} -y \\ x \\ e^z \end{bmatrix}.$$

Let K be the graph of the function $g(x, y, z) = (x^2 + y^2)^3$ over the unit circle, i.e.

$$K = \left\{ < x, y, z > \mid x^2 + y^2 \leq 1, z = (x^2 + y^2)^3 \right\}$$



The surface K.

Calculate the surface integral *of the curl of* $\mathbf{v}$ over K, i.e.

$$\iint_K \operatorname{curl} \mathbf{v} \cdot \mathrm{d}\mathbf{S}.$$

The surface normal of K is thereby assumed to point downwards.

Problem 7 **(6p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x, y) = \begin{cases} \frac{(x-y)(x^2-y^2)}{x^2+y^2} & \text{if } \langle x, y \rangle \neq \langle 0, 0 \rangle \\ 0 & \text{else.} \end{cases}$$

(a) Do the partial derivatives of f exist in $\langle 0, 0 \rangle$? (2p)

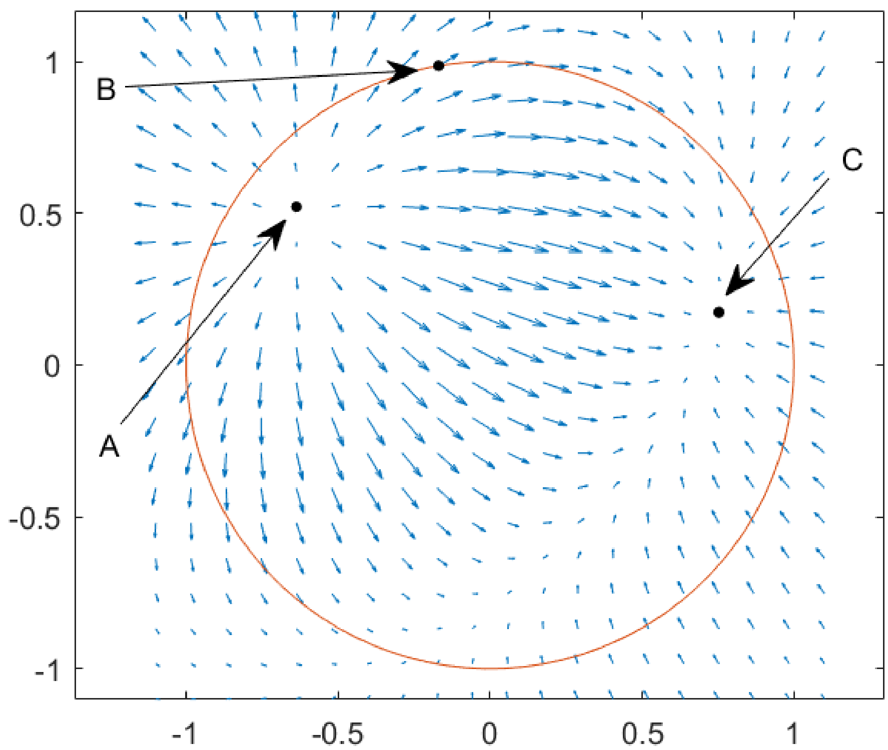
(b) Does $D_{\langle 1,1 \rangle} f(0, 0)$, i.e. the directional derivative of f in direction $\langle 1, 1 \rangle$ in the point $\langle 0, 0 \rangle$, exist? (2p)

(c) Is f differentiable in $\langle 0, 0 \rangle$? (2p)

Tip: Remember the definition of the partial and directional derivatives.

Problem 8 **(6p)**

The following figure depicts a quiver plot of the gradient ∇f of a twice differentiable function. The quiver plot is dense enough, so that the value of ∇f is not changing dramatically between the base points of the vectors. The unit circle and three points A, B and C are marked.



∇f is equal to zero in the points A and C.

(a) In which of the points A, B and C is the value of f the highest? (1p)

(b) Can B be a solution to the optimization problem (2p)

$$\max f(x, y) \text{ subject to } x^2 + y^2 = 1?$$

(c) The determinant d of the Hessian $f''(A)$ in the point A is not zero. What the sign of d? (3p)