Tentamen MVE255

Mathematical sciences, Chalmers University of Technology

Datum:	October 27th 2020, 14.00
Examiner:	Axel Flinth
Allowed aids:	Any.
Grade limits:	20 points for the grade 3. 30 points for the grade 4 40 points for the grade 5.

There are in total 50 points to collect.

Calculations and arguments should be presented in full. **Only providing an answer will normally not be rewarded with points.** Solutions may be written in Swedish or English (or German).

If you use any external tool or resource, you should reference it. See the Canvas Page for more information. Not referencing properly may result in point deduction.

The exam consists of **eight (8)** problems. They are distributed over **four (4)** pages.

Good luck!

SOLUTIONS

No guarantee of correctness!

Problem 1 **(6p)**

Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be defined through

$$g(x, y, z) = \begin{bmatrix} x^2y + e^z \\ e^z xy \end{bmatrix}.$$

(a) Calculate the derivative of g. (4p)

(b) Evaluate g and g' in $\langle 1, 0, 1\rangle$. (2p)

Solution. (a) The derivative is given by the matrix formed when writing the three partial derivatives next to each other

$$g'(x, y, z) = \begin{bmatrix} \frac{\partial g}{\partial x}(x, y, z) & \frac{\partial g}{\partial y}(x, y, z) & \frac{\partial g}{\partial z} \end{bmatrix}.$$

Thus

$$g'(x,y,z) = \begin{bmatrix} 2xy & x^2 & e^z \\ ye^z & xe^z & xye^z \end{bmatrix},$$

(b) We now simply need to insert $x = 1$, $y = 0$ and $z = 1$ into our formulas. We get

$$g(1,0,1) = \begin{bmatrix} 1^2 \cdot 0 + e^1 \\ e^1 \cdot 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} e \\ 0 \end{bmatrix}$$
$$g'(1,0,1) = \begin{bmatrix} 2 \cdot 1 \cdot 0 & 1^2 & e^1 \\ 0 \cdot e^1 & 1 \cdot e^1 & 1 \cdot 0 \cdot e^1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & e \\ 0 & e & 0 \end{bmatrix}.$$

□

Problem 2 **(6p)**

Calculate $\iiint_S z \, \mathrm{d}x\mathrm{d}y\mathrm{d}z$, where S is the part of the set

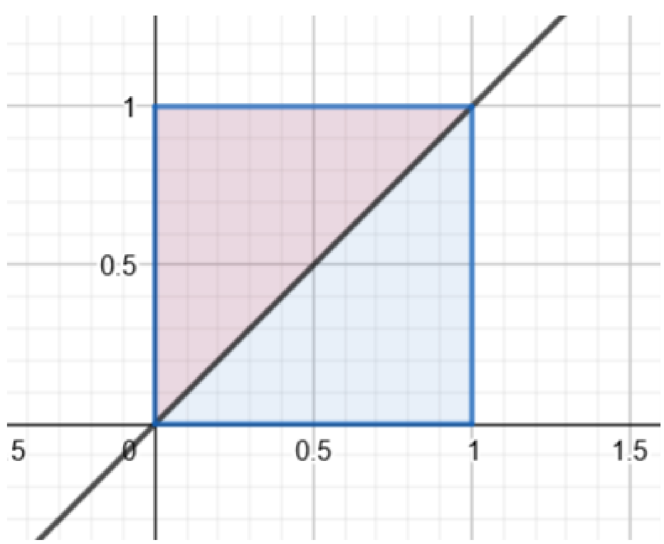
$$\{\langle x,y,z\rangle \mid 0 \le x \le y \le 1, 0 \le z \le 1\}$$

which lies above the graph of $f(x,y) = xy$.

Solution. S is the area between the graphs of $f(x,y) = xy$ and $g(x,y) = 1$ on the domain $D = \{\langle x,y\rangle \mid 0 \le x \le y \le 1\}$. Hence, by the theorem of Fubini

$$\iiint_S z \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_D \int_{xy}^1 z \, \mathrm{d}z\mathrm{d}x\mathrm{d}y = \iint_D \left[\frac{z^2}{2}\right]_{z=xy}^{z=1} \mathrm{d}x\mathrm{d}y = \iint_D \frac{1 - x^2y^2}{2} \mathrm{d}x\mathrm{d}y$$

Now we need to determine limits for D. Looking at a sketch of the domain, we see that the y-values range from 0 to 1, and that for each y, x ranges from 0 to y.



Thus,

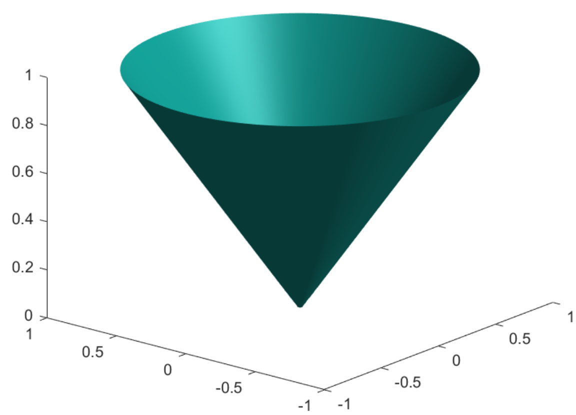
$$\iint_D \frac{1-x^2y^2}{2}\mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^y \frac{1-x^2y^2}{2}\,\mathrm{d}x\mathrm{d}y = \int_0^1 \left[\frac{x}{2} - \frac{x^3y^2}{6}\right]_{x=0}^{y} \mathrm{d}x\mathrm{d}y$$
$$= \int_0^1 \frac{y}{2} - \frac{y^5}{6}\,\mathrm{d}y = \left[\frac{y^2}{4} - \frac{y^6}{36}\right]_0^1 = \frac{2}{9}$$

□

Problem 3 **(6p)**

Calculate $\iiint_K z\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where K is the cone $\{\langle x,y,z\rangle \mid x^2+y^2 \le z^2 \le 1,\ z \ge 0\}$.

Solution. The cone is simple to describe in cylindrical coordinates: $0 \le r \le 1$, $r = \sqrt{x^2+y^2} \le z \le 1$ and $0 \le \theta \le 2\pi$ (since we make a full rotation of the z-axis).



Remembering that the functional determinant of the cylindrical coordinates is given by r, we arrive at

$$\iiint_K z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^{2\pi} \int_r^1 zr\,\mathrm{d}z\mathrm{d}\theta\mathrm{d}r = \int_0^1 \int_0^{2\pi} r\left[\frac{z^2}{2}\right]_{z=1}^{1} \mathrm{d}\theta\mathrm{d}r = \int_0^1 \pi r(1-r^2)\,\mathrm{d}r$$
$$= 2\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{1} = \frac{\pi}{4}.$$

□

Problem 4 **(8p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined through and

$$f(x,y) = e^{-x^2}e^{y^2}$$

(a) Find and classify all critical points of the function on $\mathbb{R}^2$. (4p)

(b) Find the maximum and minimum of the function on the closed unit disc (4p)

$$D = \left\{\langle x, y\rangle \mid x^2 + y^2 \leq 1\right\}.$$

Solution. (a) Critical points are by definition points where the gradient of the function is equal to zero. Let's therefore begin by calculating ∇f

$$\nabla f(x, y) = \begin{bmatrix} -2xe^{-x^2}e^{y^2} \\ 2ye^{-x^2}e^{y^2} \end{bmatrix} = e^{-x^2}e^{y^2}\begin{bmatrix} -2x \\ 2y \end{bmatrix}.$$

This only vanishes if $< x, y >=< 0, 0 >$.

To classify the the point, the canonical thing to do is to analyse the second derivative.

$$f''(x, y) = (\nabla f)'(x, y) = \begin{bmatrix} -2e^{-x^2}e^{y^2} + 4x^2e^{-x^2}e^{y^2} & -4xye^{-x^2}e^{y^2} \\ -4xye^{-x^2}e^{y^2} & 2e^{-x^2}e^{y^2} + 4y^2e^{-x^2}e^{y^2} \end{bmatrix} \implies f''(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}.$$

This matrix has a negative determinant. Therefore, the point is a saddle.

Another way to see that the point is a saddle is simply to see that for any non-zero values of x and y,

$$f(x, 0) = e^{-x^2} < 1 = f(0, 0) < e^{y^2} = f(0, y).$$

Thus, in each small neighbourhood of the origin, we find both points where f is larger and where f is smaller than in the origin. Thus, we have a saddle point.

(b) We know that the function has a saddle in the origin, and that this is the only critical point in $\mathbb{R}^2$, and therefore also on the interior of unit disc. The maximum and minimum must therefore be attained on the boundary of the disc. As such, the points will be solutions of the constrained optimization problem

$$\max f(x, y) \text{ subject to } g(x, y) = x^2 + y^2 - 1 = 0.$$

The gradient of g is equal to $< 2x, 2y >$. That is never zero for points on the unit circle, so the Lagrange theorem applies: in the point where the function is maximal, there must exist a λ so that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$e^{-x^2}e^{y^2}\begin{bmatrix} -2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

This system of equations is equivalent to

$$2x(-e^{-x^2}e^{y^2} - \lambda) = 0$$
$$2y(e^{-x^2}e^{y^2} - \lambda) = 0.$$

The first equation shows that either $x = 0$ or $-e^{-x^2}e^{y^2} - \lambda = 0$. In the first case, we get $y = \pm 1$ from the constraint. In the second case, we insert the formula for λ in the second equation to get

$$4e^{-x^2}e^{y^2}y = 0,$$

so that $y = 0$ in this case. The constraint shows that $x = \pm 1$ in this case.

Our for candidate solutions are thus $\langle 0, 1\rangle, \langle 0, -1\rangle, \langle 1, 0\rangle, \langle -1, 0\rangle$. By comparing function values, we see that the maximum occurs in $\langle 0, 1\rangle$ and $\langle 0, -1\rangle$, and is equal to e.

An alternative solution is to notice that for each point $\langle x, y\rangle$ on the unit circle, $y^2 \leq x^2 + y^2 = 1$. Therefore

$$f(x, y) = e^{-x^2}e^{y^2} \leq e^{y^2} \leq e^1 = f(0, 1) = f(0, -1).$$

This inequality is the definition of $\langle 0, 1\rangle$ and $\langle 0, -1\rangle$ being solutions of the constrained maximization problem. Along the same lines, $-x^2 \geq -x^2 - y^2 = -1$ on the unit circle, so that

$$f(x, y) = e^{-x^2}e^{y^2} \geq e^{-x^2} \leq e^{-1} = f(1, 0) = f(-1, 0).$$

which means that $\langle 1, 0\rangle$ and $\langle -1, 0\rangle$ are solutions of the constrained minimization problem.

Yet another route is to notice that on the unit circle, $x^2 = 1 - y^2$. Thus, we might as well optimize the function

$$f(x, y) = e^{x^2}e^{-y^2} = e^{1-2y^2}$$

on the closed unit interval $[-1, 1]$ (since these are the possible values for y), using methods from 1D-calculus (let us skip writing down the actual calculations, important is to not forget the boundary points $y = \pm 1$.

Comment: An elegant strategy which shortens the calculations for the entire problem is to notice that

$$f(x, y) = e^{y^2 - x^2},$$

and that the exponential is monotone. This means that the maximizers /minimizers of f are the same as the maximizers/minimizers of

$$\tilde{f}(x, y) = y^2 - x^2.$$

Applying any of the calculations in this problem to this function leads to a slightly simpler calculation.

$\square$

Problem 5 **(6p)**

If an object is thrown from ground level with an initial velocity v and an elevation angle α, the object will follow a parabola and land after (disregarding wind resistance etc.)

$$\ell = \frac{v^2 \sin(2\alpha)}{g} \text{ m.}$$

$g = 9.8$ m/s^2 is the gravity factor.

A discus thrower is known to release the discus at a speed $v = 18$ m/s and an angle $\alpha = \pi/4$ on average. The speed can deviate up to 1 m/s, and the angle $\pi/12$ radians. Use the Schrankensatz and the above formula to give an estimate of where the discus will land, with error bounds.

Use the formula. Hence, ignore both wind resistance etc. and the fact that the hand of the discus thrower is above ground level at the time of release.

Solution. Let $\tilde{v}$ and $\tilde{\alpha}$ be the actual velocity and angle. We then now that $|\tilde{v} - 18| \leq 1$ and $\left|\tilde{\alpha} - \frac{\pi}{4}\right| \leq \frac{\pi}{12}$. The Schrankensatz therefore implies

$$|\ell(\tilde{v}, \tilde{\alpha}) - \ell(18, \pi/4)| \leq M_v \, |\tilde{v} - 18| + M_\alpha \, |\alpha - \pi/4| \leq M_v + M_\alpha \frac{\pi}{12}.$$

where

$$M_v = \max_{|v-18|\leq 1, |\alpha - \pi/4| \leq \pi/12} \left| \frac{\partial \ell}{\partial v}(v, \alpha) \right| \quad M_\alpha = \max_{|v-18|\leq 1, |\alpha - \pi/4| \leq \pi/12} \left| \frac{\partial \ell}{\partial \alpha}(v, \alpha) \right|.$$

We have

$$\frac{\partial \ell}{\partial v}(v, \alpha) = \frac{2v \sin(2\alpha)}{g}$$

This is maximized in modulus when $v = 19$, and when $\alpha = \frac{\pi}{4}$, since sin is maximized in when $2\alpha = \pi/2$. Thus

$$M_v = \frac{38}{g}.$$

As for the other constant, we have

$$\frac{\partial \ell}{\partial \alpha}(v, \alpha) = \frac{-2v^2 \cos(2\alpha)}{g}.$$

This function has its maximal *absolute* value when $v = 19$ and when $\alpha = \pi/4 + \pi/6$ (the modulus of cos is minimal when $2\alpha = \pi/2$, will get larger all the way up to $2\alpha = \pi/2 + \pi/3$. Therefore

$$M_\alpha = \frac{2 \cdot 19^2 \left|\cos(\frac{2\pi}{3})\right|}{g} = \frac{2 \cdot 19^2 \cdot 1}{2g}.$$

All in all

$$\ell(\tilde{\alpha}, \tilde{v}) = \ell(18, \pi/4) \pm \left(\frac{38}{g} \cdot 1 + \frac{2 \cdot 19^2 \cdot 1}{2g} \cdot \frac{\pi}{12} \right) \approx 33.06 \pm 13.5m$$

where we in the final line used the value $g = 9.8$. $\square$

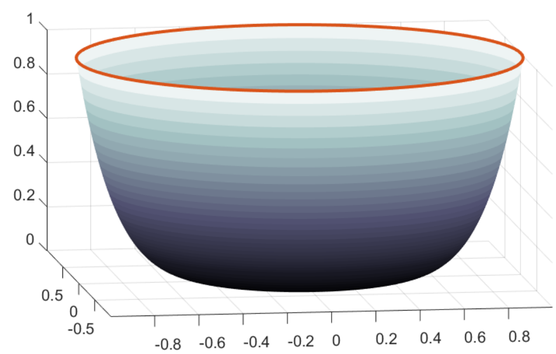
Problem 6 **(6p)**

Consider the vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{v}(x, y, z) = \begin{bmatrix} -y \\ x \\ e^z \end{bmatrix}.$$

Let K be the graph of the function $g(x, y, z) = (x^2 + y^2)^3$ over the unit circle, i.e.

$$K = \left\{ < x, y, z > \mid x^2 + y^2 \leq 1, z = (x^2 + y^2)^3 \right\}$$



The surface K.

Calculate the surface integral *of the curl of* $\mathbf{v}$ over K, i.e.

$$\iint_K \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}.$$

The surface normal of K is thereby assumed to point downwards.

Solution. We can apply the Stokes theorem:

$$\iint_K \operatorname{curl} \mathbf{v} \cdot d\mathbf{S} = \int_{\partial K} \mathbf{v} \cdot d\mathbf{s}.$$

Here, ∂K is the boundary curve of K, i.e. (as is evident from the figure provided in the problem text) the circle in the plane $z = 1$ with radius 1 and center $\langle 0, 0, 1 \rangle$. The latter curve has a parametrization

$$\gamma(\theta) = \begin{bmatrix} \cos(2\pi - \theta) \\ \sin(2\pi - \theta) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \\ 1 \end{bmatrix}, 0 \leq \theta \leq 2\pi.$$

Note that we let γ traverse ∂K in a clockwise fashion, since this harmonizes with the orientation of K. Now we can evaluate the line integral

$$\int_{\partial K} \mathbf{v} \cdot \mathrm{d}\mathbf{s} = \int_0^{2\pi} \mathbf{v}(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t = \int_0^{2\pi} \begin{bmatrix} \sin(t) \\ \cos(t) \\ e^1 \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ -\cos(t) \\ 0 \end{bmatrix} \mathrm{d}t = \int_0^{2\pi} -1\,\mathrm{d}t = -2\pi.$$

□

Problem 7 **(6p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} \frac{(x-y)(x^2-y^2)}{x^2+y^2} & \text{if } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{else.} \end{cases}$$

(a) Do the partial derivatives of f exist in $\langle 0,0\rangle$? (2p)

(b) Does $D_{\langle 1,1\rangle} f(0,0)$, i.e. the directional derivative of f in direction $\langle 1,1\rangle$ in the point $\langle 0,0\rangle$, exist? (2p)

(c) Is f differentiable in $\langle 0,0\rangle$? (2p)

Tip: Remember the definition of the partial and directional derivatives.

Solution. (a) We calculate the limits defining the partial derivatives

$$\lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(t-0)(t^2-0^2)}{t^2+0^2} - 0}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$
$$\lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(0-t)(0^2-t^2)}{0^2+t^2} - 0}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$

The limits, and therefore the partial derivatives, thus exist, and are both equal to 1.

(b) We again use the definition of the directional derivatives

$$\lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(t-t)(t^2-t^2)}{t^2+t^2} - 0}{t} = \lim_{t\to 0} \frac{0}{t} = 0$$

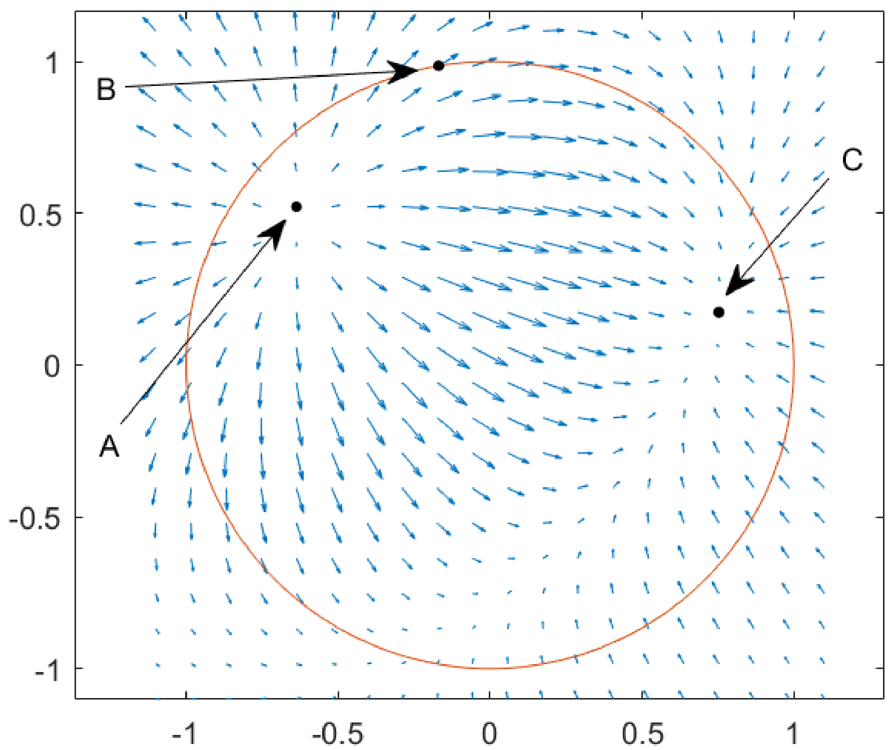
(c) No. If the function would be differentiable, we would have

$$D_{\langle 1,1\rangle} f(0,0) = \left\langle \frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right\rangle \cdot \langle 1,1\rangle = \langle 1,1\rangle \cdot \langle 1,1\rangle = 2,$$

but this is clearly not the case. □

Problem 8 **(6p)**

The following figure depicts a quiver plot of the gradient ∇f of a twice differentiable function. The quiver plot is dense enough, so that the value of ∇f is not changing dramatically between the base points of the vectors . The unit circle and three points A, B and C are marked.



∇f is equal to zero in the points A and C.

(a) In which of the points A, B and C is the value of f the highest? (1p)

(b) Can B be a solution to the optimization problem (2p)

$$\max f(x, y) \text{ subject to } x^2 + y^2 = 1?$$

(c) The determinant of the Hessian $d = \det(H''(A))$ in the point A is not zero. What the sign of d? (3p)

Solution. (a) The gradient points in the direction in which f grows. The arrows are pointing to the point C and away from the others - therefore, the function must be the largest there.

(b) No. If it where, the Lagrange criterion states that the gradient of f is parallell to the normal of the unit circle in that point, which it clearly is not.

(c) Since the gradient arrows in all directions are pointing away from A, the function grows when we deviate in any direction from A. Consequently, A must be a local minimum. Therefore, the second gradient test states that d cannot be negative – if it were, A would be a saddle. Since d also isn't zero by the problem description, we conclude that d must be positive. $\square$