Lecture 1 & 2

MVE055 / MSG810 Mathematical statistics and discrete mathematics)

Moritz Schauer

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GU & Chalmers University of Technology

Teachers

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Time table

Lecture	Monday	13-15
Exercise	Tuesday	10-12
Lecture	Wednesday	10-12
Exercise	Thursday	10-12

Course overview

`https://chalmers.instructure.com/courses/7663`

Examination

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- "För godkänd på kursen krävs godkänd på de tre grupparbetana samt godkänd på skriftlig tentamen. Betyget på kursen baseras på betyget på tentan. Tentan ger maximalt 30 poäng; för godkänd krävs åtminstone 12 poäng. För betyg 4 resp. 5 för Chalmers studenter krävs dessutom 18 resp. 24 poäng."

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- Required for passing but does not affect course grade.
- More information will be provided before the first exercise class.

Course content

In **probability theory** we construct and analyse mathematical models for phenomena that exhibit uncertainty and variation

In **statistics** we observe data and we want to infer the probabilistic model or parameters of such a model: **inverse probability.**

Generating functions allow to solve recursive equations.

The law of large number describes what happens if you perform the same experiment a large number of times.

Regression to find linear relationships between inputs/explanatory variables and outputs/explained variables.

Example:
What is the probability to throw 10 times heads in a row with a fair coin.

Example

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What is the probability to throw 10 times heads in a row with a fair coin.

This is the 10th time you throw head in a row... is that coin fair!?

Descriptive statistics

Visual inspection of data

When analysing a data set, it is a good idea to first visualise it graphically. This can also give ideas about hypotheses to test or about the relationship between variables.

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Example:
Throwing a dice 20 times we obtained the following results:

$$1, 3, 3, 3, 1, 6, 6, 5, 1, 4, 6, 1, 4, 5, 1, 1, 2, 3, 6, 5.$$

Frequency table and histogram

If the observations take values in a small set, then we can summarise the data in a frequency table showing how many outcomes we have for each possible outcome.

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For our results

$$1, 3, 3, 3, 1, 6, 6, 5, 1, 4, 6, 1, 4, 5, 1, 1, 2, 3, 6, 5,$$

we get

Outcome	1	2	3	4	5	6
Count	6	1	4	2	3	4
Proportion	0.30	0.05	0.20	0.10	0.15	0.20

Bar chart

Using the frequency table we can draw a bar chart. For each value we draw a bar whose height is proportional to the number of observations for that value.

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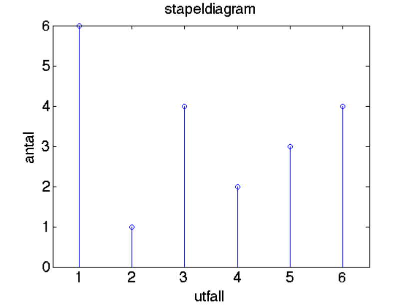

Histogram

Task: Summarise 1000 real numbers which are the outcome of some experiment,

$$12.15, 17.33, 0.96, 13.44, 11.27, 4.76, 8.26, 11.37, 24.31, 21.07, \ldots$$

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- Divide the data into a number of classes (intervals) and then calculate the number of observations in each class.

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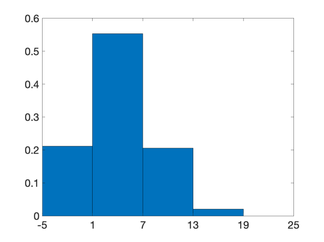
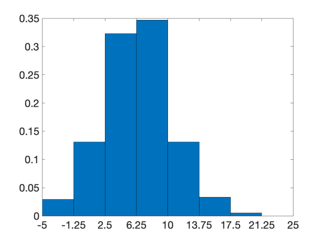
- Divide the data into a number of classes (intervals) and then calculate the number of observations in each class.
- Draw bars where the height is proportional to the number of observations in the class and the width equals the interval width.

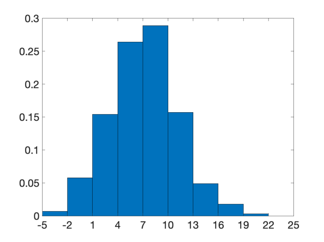
Histogram



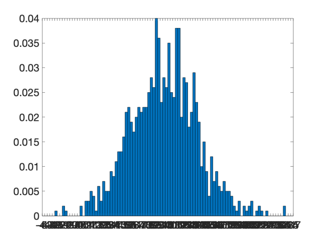
4 classes



7 classes



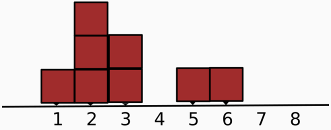
9 classes



200 classes

Sample statistics for location

Case	1	2	3	4	5	6	7	8
Value	2	3	2	6	5	1	2	3



Weights on a bar

Sample median

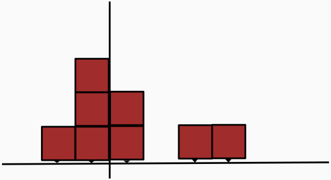
- To obtain the sample median,
  write the values in sorted order and take the middle one.

  If there is an even number of values in the data set, take the average of the two middle most.

Median

Median

Value	1	2	2	2	3	3	5	6



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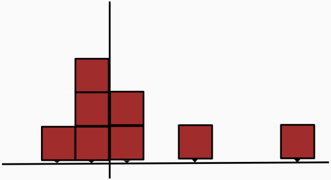

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Sample mean

- The (sample) mean, denoted as $\bar{x}$, can be calculated as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where $x_1, x_2, \cdots, x_n$ are the n observed values.

In words: Sum the values of all cases in the data set and divide by the total number of values.

Sample mean

Mean

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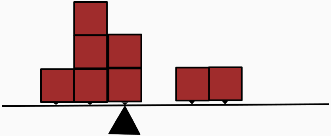

Mean $\bar{x} = \dfrac{1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 5 + 1 \cdot 6}{8} = 3$

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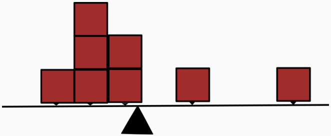

$$\text{Mean } \bar{x} = \frac{1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 5 + 1 \cdot 8}{8} = 3.25$$

Sample statistics for variation/spread

Sample variance: The sample variance of a data set $x_1, \ldots, x_n$ is given by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}((x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2)$$

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Standard deviation s: the square root $\sqrt{s^2}$ of the variance.

Example 1 (cont.)

For the dice throw example

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we obtain the mean

$$\bar{x} = (1 + 3 + 3 + \ldots + 3 + 6 + 5)/20 = 67/20 = 3.35$$

Sorting the values and taking the central one we obtain the median 3.

The variance is

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The variance is

$$s^2 = ((1 - 3.35)^2 + (3 - 3.35)^2 + \ldots + (5 - 3.35)^2)/19 = 3.8184$$

and the standard deviation is $s = 1.9541$.

Probability

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The result of the experiment is called outcome ω (*utfall*). The set of possible outcomes is called the sample space Ω (*utfallsrummet*).

Sample spaces

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We group outcomes into events.

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3. $A = \{(\text{head,head}),(\text{tail,tail})\}$, “both coins show the same face”.
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An event A occurs if any of the outcomes $\omega \in A$ occurs in the experiment.

Example

Complement

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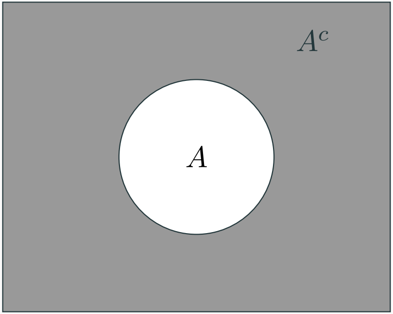

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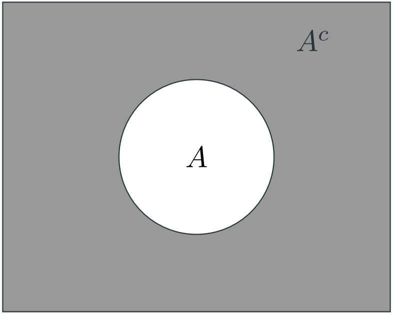

The complement of a A are all outcomes not in A.

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In the example with the die: Here $A = \{1, 3, 5\}$. So if the die shows a 2, then $A^c = \{2, 4, 6\}$ happened.

Union

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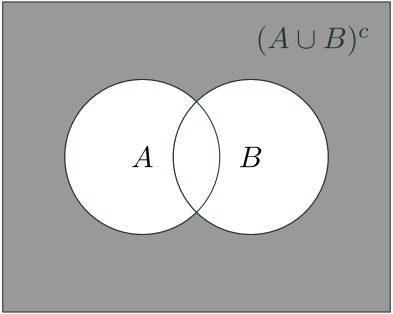

If we have events, A and B we can define $A \cup B$, the union of A and B .

- $A \cup B$ occurs if A or B occur (or both).

Intersection

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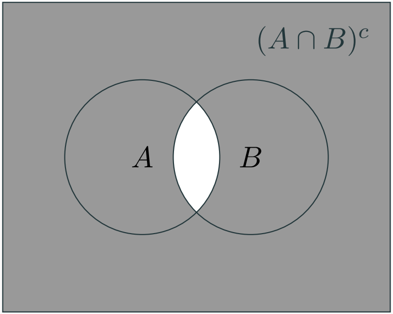

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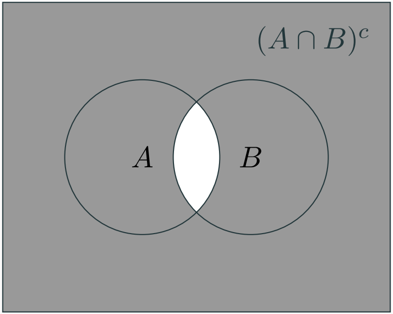

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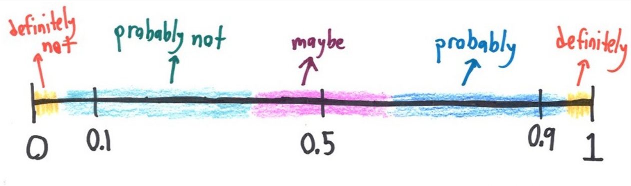
$A \cap B = \varnothing$ means that A and$B$ exclude each other.

Probabilities of events

- Probability is a numerical measure of how likely an event is to happen.



- Probability is a *proportion*, a number between 0 and 1. Notation

$$\mathrm{P}(\textit{something that can happen}) = \textit{a probability}.$$

E.g.

$$\mathrm{P}(\text{coin heads-up}) = \frac{1}{2}.$$

Figure from https://mathwithbaddrawings.com/.

Classical interpretation

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$$\mathsf{P}(A) = \frac{\text{number of outcomes for which } A \text{ occurs}}{\text{total number of outcomes}}$$

- Probability to throw an odd number with a fair die.

$$\mathsf{P}(A) = \frac{|\{1,3,5\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$$

Permutations and combinations

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Useful concepts to compute the number of possible outcomes.

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Selecting several items of without regard for their order (one of the possible ways to do this.)

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Multiplication principle

If there are a ways to make a choice and there are b ways to make a second choice, then there are ab ways to make a combined choice.

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Factorial

For $n \in \mathbb{N}$ define $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$. $n!$ is read "n-factorial".

Calculate the number of permutations

Calculate the number of selections

Theorem.)

The number of ways we can choose r objects out of a total of n distinct objects, taking into account the order, is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Calculate the number of combinations

Theorem (number of combinations.)

The number of ways we can choose r objects out of a total of n distinct objects, ignoring their order, is given by

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- $_nC_r$ is usually called binomial coefficient.

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- The frequentist interpretation of probability: Suppose we repeat a random experiment many times under identical conditions. As the number of repetitions n grows, we observe that the proportion n_A/n of times that an event A occur converges to a number. This number is the probability of A, or as formula

$$\frac{n_A}{n} \to \mathsf{P}(A), \text{where } n \to \infty$$

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Example: With a fair die, we observe the proportion of times where $A = \{\text{even number of eyes}\}$ occurs converge to $\frac{1}{2}$.

Outcome and sample space

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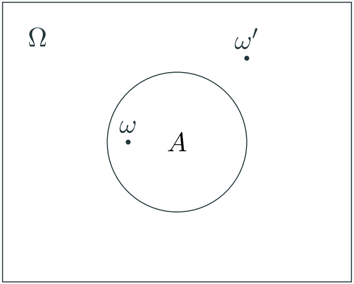
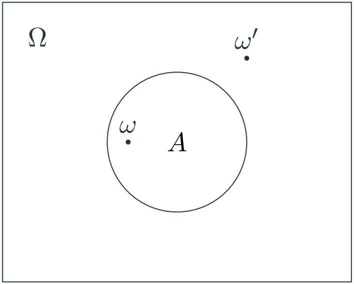
We like events because the probability of a single outcome might be too small or zero.

Event, outcome and sample space



Event A, outcome $\omega \in A$ and sample space Ω

Event, outcome and sample space



Event A, outcome $\omega \in A$ and sample space Ω

And some other outcome $\omega' \notin A$.

Intersection, union and complement

For events A and B we have defined:

Complement, A^c

Set of all outcomes ω not contained in A.

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Mutually exclusive events

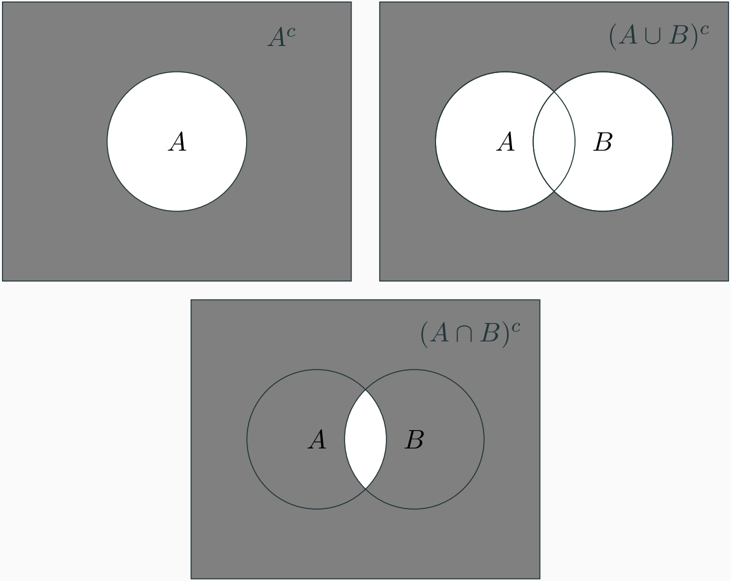
If $A \cap B = \varnothing$ then A and B are mutually exclusive events.

The n sets $A_1, A_2, \ldots, A_n$ are called mutually exclusive if

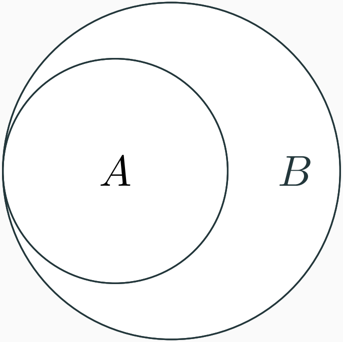
$$A_i \cap A_j = \varnothing \text{ for all } i, j,\ i \neq j.$$

Example: The set $\{2, 4, 6\}$ and the set $\{1, 3, 5\}$ are disjoint.

Venn diagram

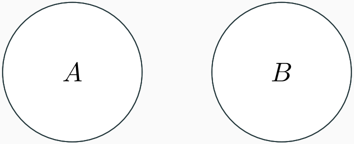


Set inclusion



$A \subset B$.

Disjoint sets



$$A \cap B = \varnothing.$$

The empty set $\varnothing$

Permutations and combinations

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Combination

A selection of objects without regard for their order.

$\{1, 3, 5\}$ is a combination of the of the numbers 1 to 6.

Note $(1, 2) \neq (2, 1)$ **but** $\{1, 2\} = \{2, 1\}$.

Permutations and combinations

Multiplication principle

If there are a ways to make a choice and there are b ways to make a second choice, then there are ab ways to make a combined choice.

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Factorial

For $n \in \mathbb{N}$ define $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$. $n!$ is read "n-factorial".

$$4! = \boxed{}$$

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Calculate the number of permutations

Theorem (number of permutations.)

The number of ways we can choose r objects out of a total of n distinct objects, taking into account the order, is given by

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Calculate the number of combinations

Theorem (number of combinations.)

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- ${}_nC_r$ is usually called binomial coefficient.

Kolmogorov's axioms

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A probability measure P is function $A \mapsto \mathsf{P}(A)$ assigning each event $A \subset \Omega$ a probability,a positive number such that

1. $0 \leq \mathsf{P}(A) \leq 1$.
2. $\mathsf{P}(\Omega) = 1$.
3. For pairwise disjoint events $A_1, A_2, \ldots$

$$\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathsf{P}(A_i).$$

Especially for disjoint/mutually exclusive events A and B,

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B).$$

Properties of probability distributions

The axioms determine all further properties of probabilities...

Properties

For the probability measure P it holds that:

1. $\mathsf{P}(\varnothing) = 0$.
2. $\mathsf{P}(A^c) = 1 - \mathsf{P}(A)$.
3. $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$.

All these properties can be seen with the help of Venn diagrams.

Probability of the union of non-disjoint events

What is the probability of drawing a jack or a red card from a well shuffled full deck (52 cards)?

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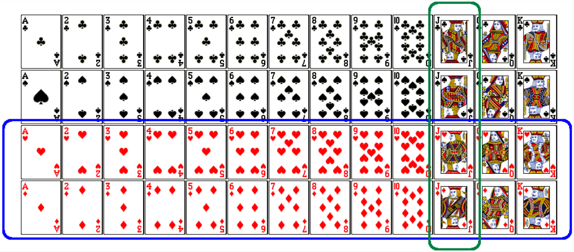

$$\mathrm{P}(\text{jack or red}) = \mathrm{P}(\text{jack}) + \mathrm{P}(\text{red}) - \mathrm{P}(\text{jack and red})$$
$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

Figure from `http://www.milefoot.com/math/discrete/counting/cardfreq.htm`.

General addition rule

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Combined experiment

Throw a coin ($\textcircled{1}$, ), and throw a 6 sided die. What is

$$\mathrm{P}(\textcircled{1}, ⚃) = \boxed{}$$

Use multiplication rule and the classical approach.

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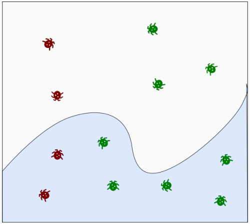
	⚀	⚁	⚂	⚃	⚄	⚅	
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

The table also shows the *marginal probablities.*

Example with the bugs

Drawing a random bug out of the aquarium, with (g)reen and (r)ed bugs on (l)and and (w)ater.



	L	W	
R	2	2	4
G	3	5	8
	5	7	12

	L	W	
R	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
G	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{2}{3}$
	$\frac{5}{12}$	$\frac{7}{12}$	1

Frequency table and probability table.

Flawed reasoning

Students at an elementary school are given a questionnaire that they are required to return after their parents have completed it.

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One of the questions asked is, "Do you find that your work schedule makes it difficult for you to spend time with your kids after school" Of the parents who replied, 85% said "no".

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Based on these results, the school officials conclude that a great majority of the parents have no difficulty spending time with their kids after school.

What went wrong?

Conditional probability

The *conditional probability* of the event of interest A given condition B is calculated as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

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Multiplication rule

If A and B represent two events, then

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Note that this formula is simply the conditional probability formula, rearranged.