Lecture 4: Continuous distributions

MVE055 / MSG810 Mathematical statistics and discrete mathematics)

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Continuous distributions

Continuous distributions

Continuous random variables

A continuous random variable can assume all values in one or several intervals of real numbers, and the probability of assuming a particular value is zero.

A continuous random variable X is described by its *probability density function (pdf)* $f(x)$

$$\mathsf{P}(a \leq X \leq b) = \int_a^b f(x)\mathrm{d}x.$$

$$P(X = x) = 0$$

and

$$\mathsf{P}(a \leq X \leq b) = \mathsf{P}(a < X \leq b) = \mathsf{P}(a \leq X < b) = \mathsf{P}(a < X < b)$$

Continuous distributions

Probability density function (pdf)

A function is a probability density function (pdf) if and only if

$$f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1.$$

Example

Show that the function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

is a pdf.

$$f(x) \geq 0 \quad \checkmark.$$

$$\int_{-\infty}^{+\infty} f(t)dt = \int_{-\infty}^{a} 0\mathrm{d}t + \int_{a}^{b} \frac{1}{b-a}dt + \int_{b}^{\infty} 0\mathrm{d}t$$
$$= \int_{a}^{b} \frac{1}{b-a}\mathrm{d}t = \frac{b-a}{b-a} = 1 \quad \checkmark.$$

Continuous distributions

Cumulative distribution function

The cumulative distribution function F of a continuous distribution is

$$F(x) = \mathsf{P}(X \leq x) = \int_{\infty}^{x} f(t)\mathrm{d}t.$$

$$\mathsf{P}(a \leq X \leq b) = F(b) - F(a)$$

Example

Find cumulative distribution function for X with pdf

$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

$$F(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x \geq b. \end{cases}$$

Expected value

The expected value is an "average" outcome of a random variable.

Expected value

The expected value of a random variable is defined as

$$\mathsf{E}(X) = \begin{cases} \int\limits_{-\infty}^{\infty} x f(x) \mathrm{d}x & \text{if } X \text{ is continuous,} \\ \sum\limits_{k=-\infty}^{\infty} k f(k) & \text{if } X \text{ is discrete.} \end{cases}$$

Rules for computing expected values

For the expected value,

- $\mathsf{E}(a) = a$.
- $\mathsf{E}(aX) = a\mathsf{E}(X)$.
- $\mathsf{E}(aX + b) = a\mathsf{E}(X) + b$.
- $\mathsf{E}(X + Y) = \mathsf{E}(X) + \mathsf{E}(Y)$.

Here X and Y are two random variables and a and b are constants.

Uniform distribution

Uniform distribution

The continuous distribution with pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}.$$

is called the *uniform distribution*. Facts: $\mathsf{E}X = (a+b)/2$.

$$\mathsf{E}X = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \frac{1}{b-a} \int_a^b t \mathrm{d}t = \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2}{a-b} = (a+b)/2$$

Transformations

If we transform the random variables by a function h we have:

Theorem

$$\mathsf{E}(h(X)) = \begin{cases} \sum\limits_{k=-\infty}^{\infty} h(k)f(k), & \text{if } X \text{ is discrete,} \\ \ldots \\ \int\limits_{-\infty}^{\infty} h(x)f(x)\mathrm{d}x, & \text{if } X \text{ is continuous.} \end{cases}$$

Variance

Variance and standard deviation

Variance

The variance of a random variable is defined as

$$\mathsf{V}(X) = \mathsf{E}[(X-\mu)^2],$$

where $\mu = \mathsf{E}[X]$ is the expected value of X.

In words, this is the expected squared deviation of the mean. The variance can be calculated by

$$\mathsf{V}(X) = \begin{cases} \sum_{k=-\infty}^{\infty} (k-\mu)^2 f(k), & \text{for discrete } X \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \mathrm{d}x, & \text{for continuous } X. \end{cases}$$

Sometimes it is easiest to compute $\mathsf{V}(X) = \mathsf{E}(X^2) - \mu^2$.

The standard deviation of a random variable X is defined as $\sigma = \sqrt{\mathsf{V}(X)}$.

Rules for computing variance

For the variance

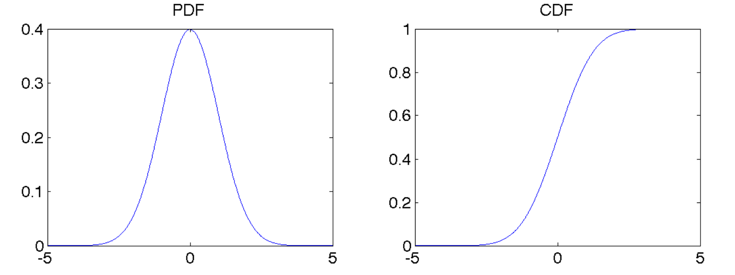
- $\mathsf{V}(a) = 0$.
- $\mathsf{V}(aX) = a^2\mathsf{V}(X)$.
- $\mathsf{V}(aX + b) = a^2\mathsf{V}(X)$.
- $\mathsf{V}(X + Y) = \mathsf{V}(X) + \mathsf{V}(Y)$, if X and Y are independent.

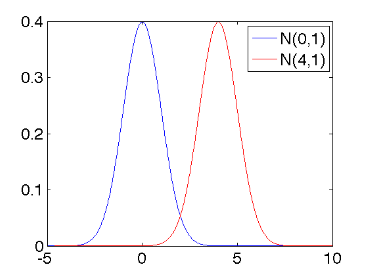
Here X and Y are two random variables and a and b are constants.

Normal distributions

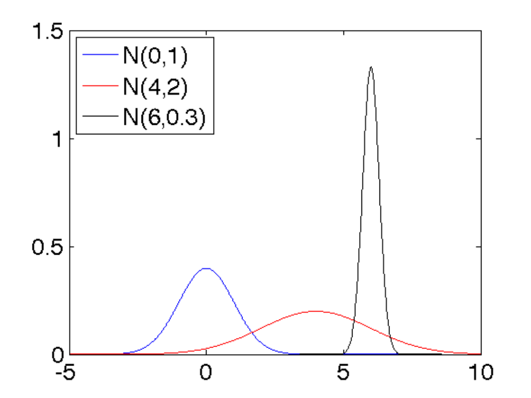
Normal distribution

Density and distribution function of $Z \sim \mathrm{N}(0,1)$ and $\mathrm{N}(4,1)$





pdf's for some other possible parameters



Normal distribution

Normal distribution $\mathsf{N}(\mu, \sigma^2)$

A continuous X is normally distributed, $\mathsf{N}(\mu, \sigma^2)$, with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, if it has pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

The distribution function is

$$F(x) = \int_{-\infty}^{x} = \ldots \text{has no nice solution}$$

Parameters

If $X \sim \mathsf{N}(\mu, \sigma^2)$ then $\mathsf{E}(X) = \mu$ and $\mathsf{V}(X) = \sigma^2$.

Normal distribution pdf

Standard normal distribution

Standard normal distribution

A continuous random variable Z is standard normally distributed if $Z \sim \mathsf{N}(0, 1)$. $\mathsf{E}[Z] = 0$ and $\mathrm{Var}(Z) = 1^2$.

We denote pdf and cdf by $\varphi(x)$ and $\Phi(x)$

Normalisation

Theorem

If $X \sim \mathsf{N}(\mu, \sigma^2)$ then $aX + b \sim \mathsf{N}(a\mu + b, a^2\sigma^2)$.

That means for $X \sim \mathsf{N}(\mu, \sigma^2)$ that

- $X = \mu + \sigma Z$ where $Z \sim \mathsf{N}(0, 1)$.
- $Z = (X - \mu)/\sigma \sim \mathsf{N}(0, 1)$.

We use this to sample random variables, and to compute probabilities:

$$\mathsf{P}(X < x) = \mathsf{P}\left(\frac{X - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right) = \mathsf{P}\left(Z < \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right).$$

Relict of the past: Normal distribution table

Table gives $\Phi(z) = \mathsf{P}(X \leq z)$ for $Z \sim \mathsf{N}(0, 1)$.
For negative values use that $\Phi(-z) = 1 - \Phi(z)$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0 :	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1 :	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2 :	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3 :	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4 :	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5 :	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6 :	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7 :	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8 :	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9 :	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0 :	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1 :	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2 :	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3 :	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4 :	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5 :	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6 :	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7 :	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8 :	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9 :	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0 :	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1 :	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2 :	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3 :	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4 :	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5 :	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952