Lecture 10: Hypothesis tests

MVE055 / MSG810 Mathematical statistics and discrete mathematics)

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Hypothesis tests

Hypothesis tests

An important problem in statistics is to test whether a theory or a *research hypothesis* is true.

Examples of such problems include:

- Does a new drug have any effect? Mean effect > 0
- Do smokers die sooner than non-smokers? Mean life time difference < 0
- Does the measuring device have a systematic error? Mean measurement error $\neq 0$

Hypothesis tests

Answers the statistical analysis could give are

1. that the research hypothesis is supported by the data (and possibly a quantification of the degree of support)
2. that the data doesn't support the hypothesis.

Example

The length of a certain lumber from a national home building store is supposed to be 2.5 m.

A builder wants to check whether the lumber cut by the lumber mill has a mean length different smaller than 2.5 m.

A statistical formulation of this problem is that we want to test the null hypothesis

$$H_0\colon \text{mean length} = 2.5\,m$$

against the alternative/research hypothesis

$$H_1\colon \text{mean length} < 2.5\,m$$

H_1 is actionable knowledge. If H_1 is true she needs to write an angry letter.

Example

- You want to test how a new employee uses laboratory equipment and therefore ask her to measure the chlorine content in a water sample $n = 5$ times.
- Results of the measurement are $\bar{x} = 59.62$ and $s^2 = 4.6920$.
- We know the true content 60, and we can assume that the measurements are samples of a random variable $X \sim \mathsf{N}(\mu, \sigma^2)$.
- The question now is whether we can claim that the new employee has a systematic error in her measurements, $\mu \neq 60$.

Setup

A statistical formulation of this problem is that we want to test the null hypothesis

$$H_0 \colon \mu = 60$$

against the alternative hypothesis or research hypothesis

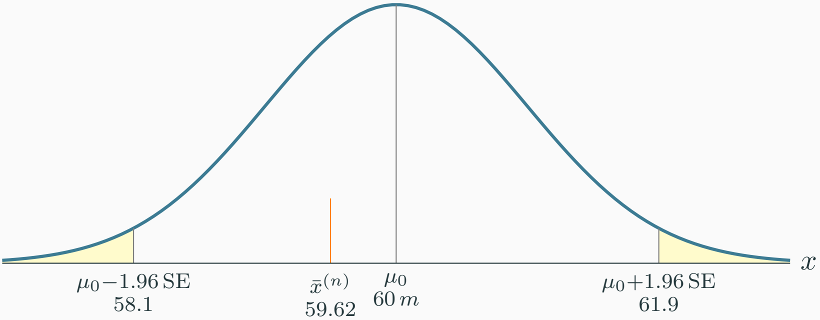
$$H_1 \colon \mu \neq 60.$$

If the test we perform finds that there is a systematic error, H_0 is rejected in favour of H_1. It is also said that μ is significantly different from 60.

Is H_1 actionable knowledge?

Choosing the alternative H_1

Choose H_1 such if someone would tell you it is true, you can do something useful with that knowledge!



$$\mathrm{SE} \approx \frac{\sqrt{4.6920}}{\sqrt{5}}$$

Decisions

The outcome of a hypothesis test can be:

- Reject H_0 (accept H_0.)
  - Action!
- Do not reject H_0
  - Could be lack of data, or H_0 being correct. The question of H_0 or H_1 is truly left open. Meh. Should still report it though.

Decision errors

		Decision	
		fail to reject H_0	reject H_0
Truth	H_0 true	✓	Type 1 Error
	H_1 true	Type 2 Error	✓

- A Type 1 Error is rejecting the null hypothesis when H_0 is true. We want to avoid that, control the probability for this error.
- A Type 2 Error is failing to reject the null hypothesis when H_1 is true.

Burden of proof

If we again think of a hypothesis test as a criminal trial then it makes sense to frame the verdict in terms of the null and alternative hypotheses:

$$
\begin{aligned}
H_0 &: \text{ Defendant is innocent} \\
H_1 &: \text{ Defendant is guilty}
\end{aligned}
$$

Which type of error is being committed in the following circumstances?

- Declaring the defendant innocent when they are actually guilty

  Type 2 error

- Declaring the defendant guilty when they are actually innocent

  Type 1 error

Which error do you think is the worse error to make?

Statistical reasoning

Classical logic: If the null hypothesis is correct, then these data can not occur.
These data have occurred.
Therefore, the null hypothesis is false.

Tweak the language, so that it becomes probabilistic... Statistical reasoning:

If the null hypothesis is correct, then these data are highly unlikely.
These data have occurred.
Therefore, the null hypothesis is unlikely.

Definition

In statistical hypothesis testing, a result has statistical significance when it is very unlikely to have occurred given the null hypothesis.

The significance level α is the (tolerated) probability of making a type I error:

About failure to reject H_0

If you want to take a decision in the case the test fails to reject H_0, you should compute the type II error probability first. This is typically difficult.

Therefore we should avoid far reaching decisions if our tests fail to reject H_0.

Tests from confidence intervals

Data (samples from a distribution with unknown parameter μ).

Hypothesis about parameter. Here $H_0 : \mu = \mu_0$ and $H_1 \colon \mu \neq \mu_0$.

Significance level α, e.g $\alpha = 5\%$.

Decision rule: Compute a $(1 - \alpha)(= 95\%)$-confidence interval $[A, B]$ for the parameter μ. If the $\mu_0 \notin [A, B]$, reject H_0.

Type 1 error: This rule has type 1 error of $5\,\%$, so this is a valid test for level $\alpha = 5\,\%$.

Tests with test statistics

Data (samples with unknown population parameter μ).

Hypothesis about parameter. Here $H_0 : \mu = \mu_0$ and $H_1 : \mu \overset{\neq}{\underset{<}{>}} \mu_0$.

Significance level α, e.g $\alpha = 5\%$.

Test statistic T: Typically, T comes from an estimator for our parameter with known distribution under H_0.

$$T = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \quad \text{(example)}$$

Decision rule: Reject H_0 if the p-value is less than the significance level α.
or: Reject H_0 if the T_{obs} is in the critical region/rejection region (see next slide).

Type I error: The type I error for this test is $\leq \alpha$.

Critical region

The critical region C_α of a test are those values of the test statistic T for which H_0 can be rejected while obeying significance level α. Typically represented by one or two critical values.

We compute rejection region for the data. We reject H_0 if T_{obs} is in the rejection region.

Example: Critical values for mean of normal population

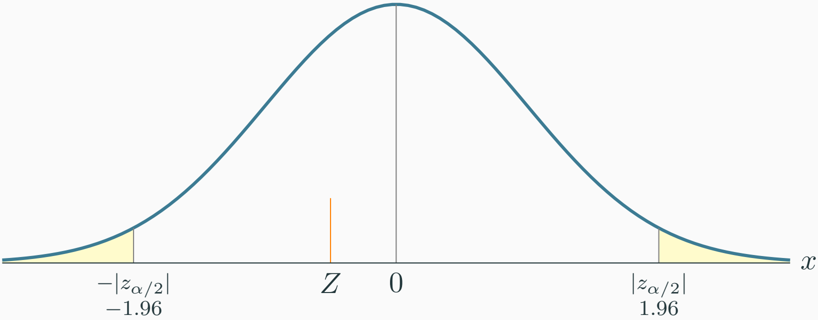
We want to use a quantity T that we know the distribution of under H_0, so that we can calculate the p-value.

In case of the normal distribution with known variance

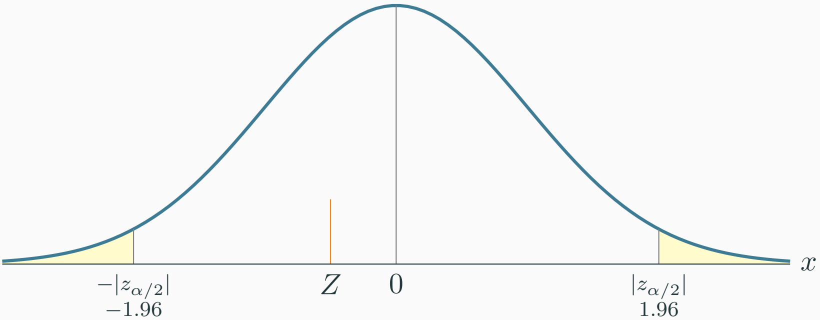
$$(T =)Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

we know that Z under H_0 is $\mathsf{N}(0, 1)$-distributed and

Reject H_0 at level α if $|Z| > z_{\alpha/2}$.



Rejection region for $\alpha = 0.05$.



Rejection region for $\alpha = 0.05$ (on the x-axis below the yellow area).

p-value

The p-value of the test is defined as the probability under the null hypothesis that we get a value T which is at least as "extreme" as the observed value T_{obs}.

Example: p-value for normal distribution

We want to use a quantity T that we know the distribution of under H_0, so that we can calculate the p-value.

In case of the normal distribution with known variance

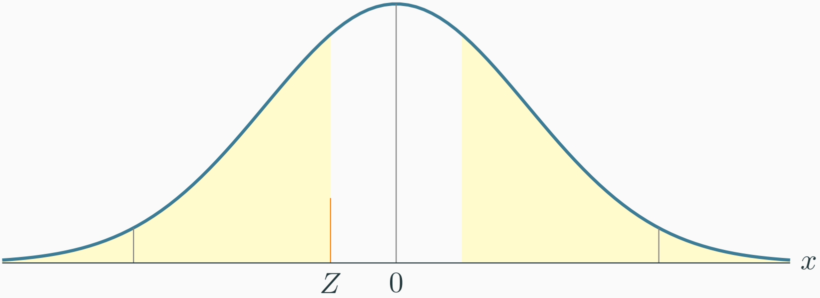
$$T = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

we know that T under H_0 is $\mathsf{N}(0, 1)$-distributed and

$$p = \mathsf{P}(|T| \geq |T_{obs}|) = 2 \cdot \mathsf{P}(T \geq |T_{obs}|) = 2(1 - \Phi(|T_{obs}|)).$$

We compute p for the data. We reject H_0 if $p < \alpha$

We compute rejection region for the data. We reject H_0 if T_{obs} is in the rejection region.



Yellow area: p value.

How many observations are needed?

A test detects a deviation of $\mu - \mu_0$ more easily if:

- If the significance level α is not very small.
- The number of observations n is large.
- The population variance relatively σ^2 is small.