FUNCTIONS

A *function* or *map* is often seen as a rule that associates, to each element of a set A, exactly one element of a given set B. Formally, this can be expressed as follows:

> Let A and B be sets. A *function from A to B* is a set f of ordered pairs of elements (a, b), where $a \in A$ and $b \in B$, so that each element $a \in A$ belongs to *exactly one* of the pairs in f.

If f is a function from A to B we denote this by

$$f : A \to B$$

We call A the *domain* of f and B the *range* of f.

We often see a function f as a machine that, when we input an element $a \in A$, outputs an element $b \in B$, namely the element that f associates to a. This is denoted as follows:

$$f(a) = b.$$

Example: Let A be a set of three persons, which we call a, b, c. Let f be the function that to each person (element of) A associates her height (in cm). Then f can be regarded as a function $f : A \to \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers. If the respective heights of a, b, c are 170, 160 and 180, then the set f consists of the pairs

$$f = \{(a, 170),\ (b, 160),\ (c, 180)\}.$$

A different, and much more common, way of expressing this is as follows:

$$f(a) = 170, \qquad f(b) = 160, \qquad f(c) = 180.$$

A well known function $f : \mathbb{Z} \to \mathbb{N}$ (where $\mathbb{Z}$ is the set of all integers) is the function that to each integer associates its square. This function is usually described by the equation

$$f(n) = n^2.$$

Observe that if we say that the square root of a number n is a number whose square is n, then we have *not* described a function, since we would be associating both 2 and -2 to 4. In order to define the square root of a number as a function we therefore usually decide to take the non-negative square root. Thus, we set $\sqrt{9} = 3$ (and not ± 3) and then we can regard $g(x) = \sqrt{x}$ as a function.

A function $f : A \to B$ is *injective* if it sends no two different elements in A to the same element in B. Formally, f is injective if $f(a) \neq f(b)$ when $a \neq b$. Equivalently, f is injective if $f(a) = f(b)$ implies $a = b$. injective = one-to-one

A function $f : A \to B$ is *surjective* if it "hits" every element in B, that is, if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$. surjective = onto

A function that is both injective and surjective is said to be *bijective*.

Suppose that $f : A \to B$ and $g : B \to C$. We define the *composition of g with f*, denoted $g \circ f$, by setting

$$(g \circ f)(x) = g(f(x)).$$

Observe that $g \circ f$ is a function from A to C. In fact, it is enough for the range of f to be a *subset* of the domain of g in order for the composition to be defined. For example, if $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 3n - 2$ and $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = x/2$, then $g \circ f : \mathbb{Z} \to \mathbb{R}$ is given by $g(f(n)) = (3n - 2)/2$.

Unfortunately (for us in this part of the world), due to a historical accident, functions, when composed, are read from right to left.

If $f : A \to B$ is a bijective function, then we can define an *inverse* function to f. This inverse function is denoted f^{-1} and has the property that $f^{-1}(f(a)) = a$ and $f(f^{-1}(b)) = b$ for all $a \in A$ and for all $b \in B$.

Suppose $f : A \to B$ and let S be a subset of A (this is denoted $S \subset A$). The *restriction of f to S*, usually denoted $f|_S$, is the function with domain S and range B that has the same values on each element of S as f does. In other words, $f|_S$ is still defined by the same rule, but can now only be applied to the elements of S.

If $f : A \to B$, then the *image* of f, denoted $\operatorname{Im} f$, is the set of elements in B that are "hit" by f, that is, the set

$$\operatorname{Im} f = \{f(a) \mid a \in A\}.$$

For example, if $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(x) = 2x$, then $\operatorname{Im} f = \{0, 2, 4, 6, \ldots\}$. Observe that a function $f : A \to B$ is surjective if and only if $\operatorname{Im} f = B$.

Some Examples

If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 2$ then f is both injective and surjective. It is injective because if $f(a) = f(b)$ then $3a - 2 = 3b - 2$, so $a = b$. It is surjective, because for each $b \in \mathbb{R}$ we can find an $a \in \mathbb{R}$ such that $f(a) = b$, namely $a = b/3 + 2/3$.
Thus, f is bijective, so it has an inverse. The inverse is the function f^{-1} defined by $f^{-1}(x) = x/3+2/3$, because $f^{-1}(f(x)) = f^{-1}(3x-2)) = (3x-2)/3+2/3 = x-2/3+2/3 = x$. You should check for yourself that $f(f^{-1}(x)) = x$.

$\mathbb{R}$ is the set of all real numbers

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not surjective, because there is no $a \in \mathbb{R}$ such that $f(a) = -1$. It is not either injective, because $f(-5) = f(5) = 25$.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is both surjective and injective (why?).

bijective

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is bijective; its inverse is $f^{-1}(x) = x/2$.

The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$ is injective but *not* surjective, for there is no $a \in \mathbb{Z}$ such that $f(a) = 3$.

$\mathbb{Z}$ is the set of all integers

The function $f : \mathbb{Z} \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

is surjective, but not injective, since all even numbers are sent to 0 (and all odd numbers to 1).