Logic in Computer Science
DAT060/DIT202

Recap Exercises

Week 1

Logic

1. Let p, q, r be the following propositions: $\begin{array}{ll} p\text{:} & \text{“it is raining”} \\ q\text{:} & \text{“the sun is shining”} \\ r\text{:} & \text{“there are clouds in the sky”} \end{array}$.

   Translate the following into logical notation, using p, q, r and logical connectives.

   (a) It is raining and the sun is shining;
   (b) If it is raining then there are clouds in the sky;
   (c) If it is not raining then the sun is not shining and there are clouds in the sky;
   (d) The sun is shining if and only if it is not raining;
   (e) If there are no clouds in the sky then the sun is shining.

2. Let p, q, r be as in exercise 1). Translate the following into English sentences.

   (a) $(p \wedge q) \rightarrow r$;
   (b) $(p \rightarrow r) \rightarrow q$;
   (c) $\neg p \leftrightarrow (q \vee r)$;
   (d) $\neg(p \leftrightarrow (q \vee r))$;
   (e) $\neg(p \vee q) \wedge r$.

3. Give the truth value of the propositions in exercises 1) and 2).

4. Which of the following propositions is logically equivalent to $p \rightarrow q$:

   $\neg p \rightarrow \neg q, \qquad q \rightarrow p, \qquad \neg q \rightarrow \neg p, \qquad \neg q \vee p, \qquad \neg p \vee q, \qquad p \wedge \neg q, \qquad q \wedge \neg p.$

5. Construct the truth tables for:

   (a) $(p \rightarrow q) \rightarrow ((p \vee \neg q) \rightarrow (p \vee q))$;
   (b) $((p \vee q) \wedge r) \rightarrow (p \wedge \neg q)$;
   (c) $((p \leftrightarrow q) \vee (p \rightarrow r)) \rightarrow (\neg q \wedge p)$.

6. Suppose that $p \to q$ is known to be false. Give the truth values for

$$p \wedge q, \qquad p \vee q, \qquad q \to p.$$

7. Write down the negation of the following statement: "for every number x there is a number y such that $y < x$". Find an equivalent formulation without negation.

8. Find an equivalent formulation to $\neg \forall x.(P(x) \to Q(x))$ which does not contain a negation at the front nor an implication inside.

9. Consider the statement "everybody loves someone sometime". Let $L(x, y, z)$ be a proposition stating that x loves y at time z. Using this notation, express the original statement using quantifiers.

10. Let $F(x, y)$ be the proposition "you can fool person x at time y". Using this notation, write a quantified statement to formalise Abraham Lincoln's statement: "you can fool all the people some of the time, you can fool some people all the time, but you cannot fool all people all the time".

11. Consider the following universes:

$$(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\} \text{ and } [0, 1] = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 1\}.$$

Determine whether the statements below are true or false in each of these universes.

(a) $\forall x.\exists y.x > y$;
(b) $\forall x.\exists y.x \geqslant y$;
(c) $\exists x.\forall y.x > y$;
(d) $\exists x.\forall y.x \geqslant y$.

Sets

1. Write the following sets in enumerated form:

(a) The set of all vowels;
(b) $\{x \in \mathbb{N} \mid 10 \leqslant x \leqslant 20 \text{ and } x \text{ is divisible by } 3\}$;
(c) The set of all natural numbers that leave a remainder of 1 after division by 5.

2. Write the following sets using a characteristic property:

(a) $\{4, 8, 12, 16, 20\}$;
(b) $\{000, 001, 010, 011, 100, 101, 110, 111\}$;
(c) $\{1, 4, 9, 16, 25, \ldots\}$.

3. Let $A = \{a, b, c\}$ and $B = \{p, q\}$. Write down the following sets in enumeration form:

$$A \times B, \qquad A^2, \qquad B^3.$$

4. Let $A = \{1, \{1\}, \{2\}, 3\}$. Identify which of the following statements are true or false.

   (a) $\emptyset \in A$, $\emptyset \subseteq A$;

   (b) $1 \in A$, $1 \subseteq A$;

   (c) $\{1\} \in A$, $\{1\} \subseteq A$;

   (d) $\{\{1\}\} \subseteq A$;

   (e) $2 \in A$;

   (f) $\{2\} \in A$, $\{2\} \subseteq A$;

   (g) $\{3\} \in A$, $\{3\} \subseteq A$.

5. Let $\{x \in \mathbb{N} \mid x \leqslant 12\}$ be our universe. Let $A = \{x \mid x \text{ is odd}\}$, $B = \{x \mid x > 7\}$ and $C = \{x \mid x \text{ is divisible by } 3\}$. Write down the following sets in enumerated form:

   (a) $A \cap B$;

   (b) $B \cup C$;

   (c) $\overline{A}$;

   (d) $(A \cup \overline{B}) \cap C$;

   (e) $\overline{A \cup C} \cup \overline{C}$.

6. Show that $\overline{\overline{A} \cap B} = A \cup \overline{B}$ using the laws of sets.

7. Show that

   (a) Difference of sets is not commutative, that is, $A - B = B - A$ can fail.

   (b) Show that $A - B = B - A$ if and only if $A = B$.

   (c) Difference of sets is not associative, that is, $A - (B - C) = (A - B) - C$ can fail.

   (d) Show that $A - (B - C) = (A - B) - C$ if and only if $A \cap C = \emptyset$.

8. Prove the following properties on sets A, B, C:

   (a) $A - B = A \cap \overline{B}$;

   (b) $A \subseteq B$ if and only if $A - B = \emptyset$;

   (c) $A - (A - B) = A \cap B$;

   (d) $A \cap B \subseteq (A \cap B) \cup (B \cap \overline{C})$;

   (e) $(A \cup C) \cap (B \cup \overline{C}) \subseteq A \cup B$;

   (f) $A \cap B = \emptyset$ if and only if $A \subseteq \overline{B}$ if and only if $B \subseteq \overline{A}$;

   (g) $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$;

   (h) $(A \cup B) - (A \cup C) \subseteq B - C$;

   (i) $(A \cup B) - C = (A - C) \cup (B - C)$;

   (j) $A - (B - C) = (A - B) \cup (A \cap C)$.

Relations

1. Determine which of these relations are reflexive, symmetric, antisymmetric and transitive.

   (a) "is a sibling of", on the set of all people;

   (b) "is the son of", on the set of all people;

   (c) "is greater than", on the set of real numbers;

   (d) "has the same integer part", on the set of real numbers;

   (e) "is a multiple of", on the set of natural numbers;

   (f) The relation R on the set of real numbers defined by $x\,R\,y$ if $x^2 = y^2$.

2. Prove that logical equivalence is an equivalence relation on the set of all propositional formulas with a fixed set of atoms.

3. Let $R \subseteq \mathbb{Z}\times\mathbb{Z}$ such that $x\,R\,y$ if $x-y$ is divisible by 4. Show that R is an equivalence relation.

4. Prove, by supplying a counterexample, that no two of reflexivity, symmetry, and transitivity imply the third.

5. Let R be a relation on the set S of students at a school such that for $x, y \in S, x\,R\,y$ if and only if x and y have a class together. Determine whether R is an equivalence relation.

6. The inclusion relation $\subseteq$ is a relation on the power set $\mathcal{P}(S)$ of a set S. Write the elements of such relation for $S = \{1, 2, 3\}$.

7. Let R_1 and R_2 be relations on a set S.

   (a) Show that $R_1 \cap R_2$ is reflexive if R_1 and R_2 are;

   (b) Show that $R_1 \cap R_2$ is symmetric if R_1 and R_2 are,

   (c) Show that $R_1 \cap R_2$ is transitive if R_1 and R_2 are;

   (d) Must $R_1 \cup R_2$ be reflexive if R_1 and R_2 are?

   (e) Must $R_1 \cup R_2$ be symmetric if R_1 and R_2 are?

   (f) Must $R_1 \cup R_2$ be transitive if R_1 and R_2 are?

8. Three relations are given on the set of all nonempty subsets of $\mathbb{N}$. In each case, determine whether the relation is reflexive, symmetric and transitive. Would these answers change if we would consider the set of *all* subsets of $\mathbb{N}$. Justify.

   (a) $A\,R\,B$ if and only if $A \subseteq B$;

   (b) $A\,R\,B$ if and only if $A \cap B \neq \emptyset$;

   (c) $A\,R\,B$ if and only if $1 \in A \cap B$.

9. For the following relations on $S = \{0, 1, 2, 3\}$, write each relation as a set of ordered pairs. In addition, determine which of these relations are reflexive, symmetric, antisymmetric and transitive.

(a) $m\,R_1\,n$ if $m+n=3$;

(b) $m\,R_2\,n$ if $m \leqslant n$;

(c) $m\,R_3\,n$ if $\mathsf{max}\{m,n\}=3$;

(d) $m\,R_4\,n$ if $m-n$ is even;

(e) $m\,R_5\,n$ if $m+n \leqslant 4$.

Functions

1. Let $f : A \to B$. Show that the relation R defined as $x\,R\,y$ if $f(x) = f(y)$ is an equivalence relation.

2. Determine which of the following functions are injective and surjective.

   (a) $f : S \to S$ for S a finite set of nonempty strings which is closed under the reverse operation, and $f(s)$ the function that returns the reverse of the string s;

   (b) $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(x,y) = x+y$;

   (c) $s : \mathbb{N} \to \mathbb{N}$ such that $s(n) = n+1$;

   (d) h : {English words} $\to$ {letters} such that $h(w)$ returns the first letter of the word w;

   (e) $|\_| : \mathcal{P}(A) \to \mathbb{N}$ such that $|X|$ is the cardinality of the set $X \subseteq A$, for A any given finite set.

3. Let $f : \{1,2,3,4,5\} \to \{0,1,2,3,4,5\}$ be defined as $f(n) = 3n \bmod 5$. Determine all the pairs that are related by f. State whether f is injective and surjective.

4. Consider the following functions from $\mathbb{N}$ to $\mathbb{N}$:

$$id(n) = n, \qquad f(n) = 3n, \qquad g(n) = n + (-1)^n,$$
$$h(n) = \mathsf{min}\{n, 100\}, \qquad k(n) = \mathsf{max}\{0, n-5\}.$$

   (a) Which of these functions are injective?

   (b) Which of these functions are surjective?

5. Here are two "shift" functions mapping $\mathbb{N}$ to $\mathbb{N}$:

$$f(n) = n+1 \qquad g(n) = \mathsf{max}\{0, n-1\}$$

   (a) Calculate $f(n)$ for $n = 0,1,2,3,4,73$;

   (b) Calculate $g(n)$ for $n = 0,1,2,3,4,73$;

   (c) Show that f is injective but not surjective;

   (d) Show that g is surjective but not injective.

   (e) Show that $g \circ f(n) = n$ for all n, but $f \circ g(n) = n$ does not hold for all n.

6. Find the inverse of each of the following functions, or explain why no inverse exists.

   (a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 3x+2$;

(b) $|\_| : \mathbb{R} \to \mathbb{R}$, the absolute value of a real numbers;

(c) $g : \mathbb{N} \to \mathbb{N}$ where $g(x) = \begin{cases} n+1 \text{ if } n \text{ is odd} \\ n-1 \text{ if } n \text{ is even} \end{cases}$;

(d) $h : S \to S$ for S a finite set of nonempty strings and h the function that moves the last character of the string to the beginning, for example $h(\texttt{abcd}) = \texttt{dabc}$;

(e) $k : \mathbb{R} \to \mathbb{R}$ such that $k(x) = x^3 - 2$.

7. Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ defined by $f(x, y) = (x+y, x-y)$. Show that f is a bijection and find a formula for f^{-1}.

8. Let A, B, C, D be sets, and let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Prove that composition of functions is associative.

9. Prove that a function cannot have more than one inverse.
Hint: Assume inverses are not unique and try to deduce a contradiction using exercise 8).

10. Let $f : S \to T$ and $g : T \to U$ be invertible functions, that is, have inverse functions.

    (a) Show that f^{-1} is invertible and that $(f^{-1})^{-1} = f$;

    (b) Show that $g \circ f$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

11. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be as follows:

$$f(x) = 4x - 3, \qquad g(x) = x^2 + 1, \qquad h(x) = \begin{cases} 1 \text{ if } x \geqslant 0 \\ 0 \text{ if } x < 0 \end{cases}.$$

Find an expression defining the following functions:

    (a) $f \circ f$, $g \circ f$, $h \circ f$;

    (b) $f \circ g$, $h \circ g$;

    (c) $f \circ h$, $g \circ h$.

Mathematical and Course-of-values Induction

1. Prove that $\sum_{k=0}^{n} k = n(n+1)/2$.

2. Prove that $\sum_{k=1}^{n}(2k-1) = n^2$.

3. Prove that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

4. Prove that $\forall n \geqslant 4. n^2 \leqslant 2^n$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by recursion as

$$f(0) = 0 \qquad f(n+1) = f(n) + n$$

What are the values of $f(2)$ and $f(3)$?

Use mathematical induction to show that for all $n \in \mathbb{N}$ we have

$$2f(n) = n^2 - n$$

6. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.

7. Let us define by recursion the following function:

$$0! = 1 \qquad (n+1)! = (n+1) \times n!$$

Show that $n! \geqslant 2^n$ for $n \geqslant 4$ by analogy with the proof of example 1.17, page 21 of the main book.

8. Consider the following definitions for $f, g : \mathbb{N} \to \mathbb{N}$:

$$\begin{array}{ll} f(0) = 0 & g(0) = 0 \\ f(1) = 1 & g(n+1) = 1 - g(n) \\ f(n+2) = f(n) & \end{array}$$

Compute $f(2), f(3), g(1), g(2)$ and $g(3)$.

Prove that $\forall n \in \mathbb{N}.\ f(n) = g(n)$.

9. Let us define the Fibonacci function:

$$f(0) = 0 \qquad f(1) = 1 \qquad f(n+2) = f(n+1) + f(n)$$

We then define $s(0) = 0,\ s(n+1) = s(n) + f(n+1)$.

Prove by induction that we have

$$\forall n. s(n) = f(n+2) - 1.$$

Now we define

$$l(0) = 2,\ l(1) = 1,\ l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have $l(n+1) = f(n) + f(n+2)$.

Inductive Sets and Structural Induction

1. Consider the inductive definition of lecture 2 and of its functions len, $+\!\!+$ and rev.

   Let xs and ys be lists. Use structural induction to prove the following properties:

   (a) $\mathsf{rev}(xs +\!\!+ ys) = \mathsf{rev}(ys) +\!\!+ \mathsf{rev}(xs)$;

   (b) $\mathsf{rev}(\mathsf{rev}(xs)) = xs$;

   (c) $\mathsf{len}(\mathsf{rev}(xs)) = \mathsf{len}(xs)$.

2. A binary tree with information on the nodes is either a leaf with no information or a node containing some information and exactly 2 subtrees. You may assume the information in the nodes being of any suitable type (string, Natural numbers, ...).

   (a) Give the inductive definition of the set `Tree` of trees .

(b) Define the functions that count the number of leaves nl, the number of nodes that are not leaves nn, and the number of subtrees nt of a tree.

(c) Prove by structural induction on the trees that $\forall t \in \texttt{Tree}.\, \mathsf{nl}(t) = \mathsf{nn}(t) + 1$.

(d) Prove by structural induction on the trees that $\forall t \in \texttt{Tree}.\, 2\mathsf{nn}(t) = \mathsf{nt}(t)$.

3. Consider expressions that consist of either an identifier, the addition of two expressions, or the multiplication of two expressions. You may assume the identifiers being of any suitable type (string, Natural numbers, ...).

   (a) Give the inductive definition of the set $\texttt{Exp}$ of expressions;

   (b) Define the function op that counts the number of additions and multiplications in an expression, and the function id that counts the number of identifiers in an expression;

   (c) Prove by structural induction on the expressions that $\forall e \in \texttt{Exp}.\, \mathsf{op}(e) + 1 = \mathsf{id}(e)$.

4. Let BP be the set of strings with balanced parentheses. The simplest element in this set consists of the character '(' followed by the character ')'. Given an element x in the set, we construct a new element in this set by adding a '(' in front of x and a ')' in the back of x. Given two elements in this set, we can also form a new element by concatenating these two elements. For example, an element in this set is the string "(()())(())".

   (a) Give the inductive definition of the set BP of string with balanced parentheses;

   (b) Define the functions nrOP and nrCP that count the number of characters '(' and ')', respectively, in a string in BP.

   (c) Prove using induction that $\forall x \in \texttt{BP}.\, \mathsf{nrOP}(x) = \mathsf{nrCP}(x)$.