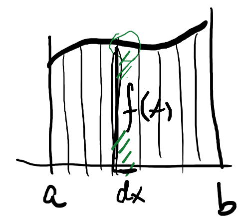
Riemann integral, grunder

den 30 september 2020 15:33

Är n antalet rektanglar du delar in det i? JA

$$A = \int_a^b f(x)\,dx \qquad \int - \text{summa}$$

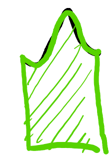
$$\int_a^b f(x)\,dx = \sum_1 \overbrace{f(x)\,dx}^{dA}$$



$$A = \sum dA = \int dA$$

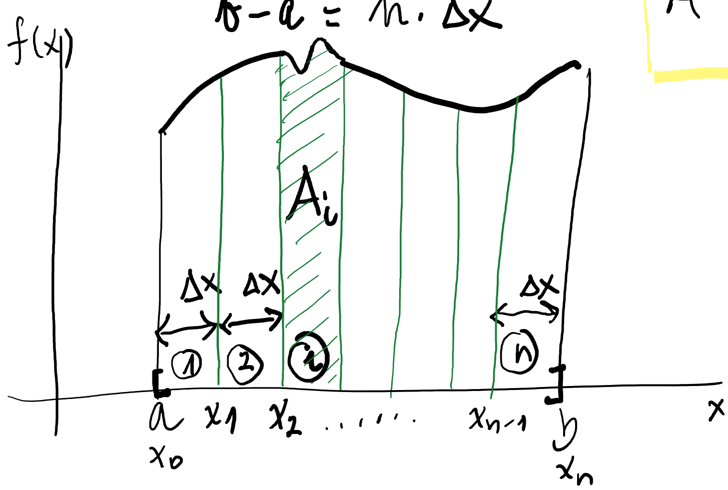
$$A_1 + A_2 + \dots + A_n$$

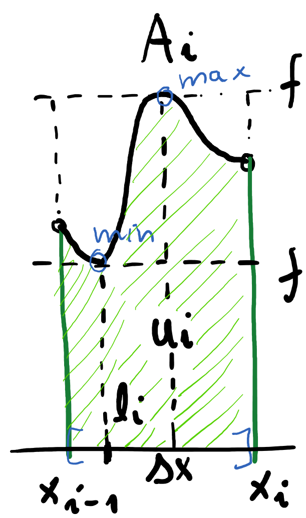
$$\boxed{A = \sum_{i=1}^{n} A_i}$$

$A_i =$ 

Integral som yta under grafen

$$b - a = n \cdot \Delta x$$





$f_i^{max} = f(u_i)$

min/max satsen

$f_i^{min} = f(l_i)$

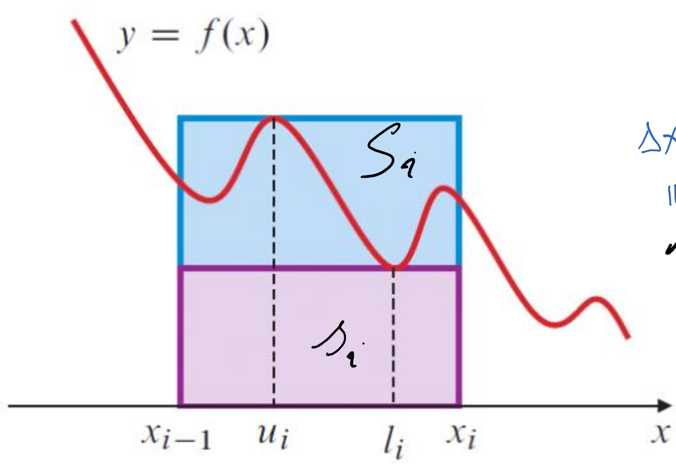
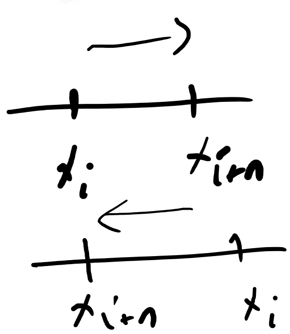


Figure 5.10

$$\underset{\parallel}{\Delta x \cdot f_i^{min}} \qquad \underset{\parallel}{\Delta x \cdot f_i^{max}}$$

$$s_i \le A_i \le S_i$$



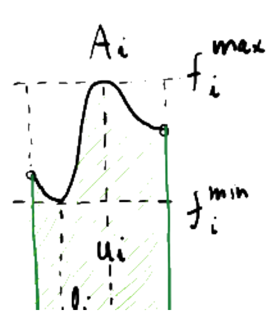
$\Delta x > 0$

$\Delta x < 0$

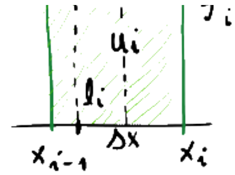
$$\boxed{\Delta x = x_{i+1} - x_i}$$

$$S_i = \Delta x \cdot f_i^{max}$$

$$s_i = \Delta x \cdot f_i^{min}$$

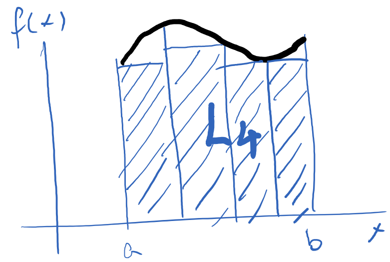


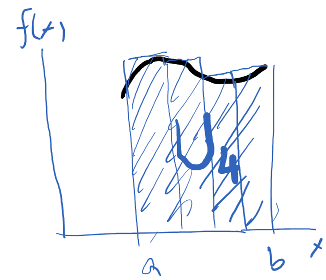
$s_i = \Delta x \cdot t_i$



$s_i \leq A_i \leq S_i \quad \Big| \sum_{i=1}^{n}$

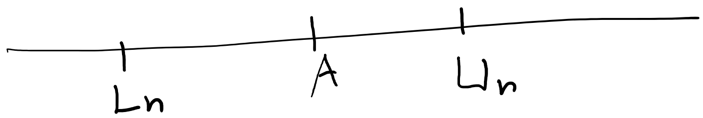
$$\underbrace{\sum_{i=1}^{n} s_i}_{L_n} \leq \underbrace{\sum_{i=1}^{n} A_i}_{A} \leq \underbrace{\sum_{i=1}^{n} S_i}_{U_n}$$



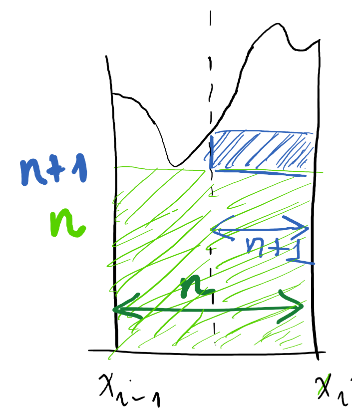


lower Riemann sum

upper Riemann sum



$L_n \leq L_{n+1}$



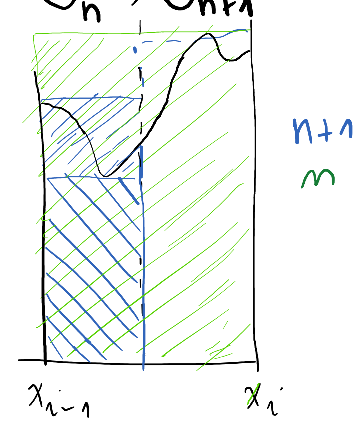
If P_1 and P_2 are two partitions of $[a, b]$ such that every point of P_1 also belongs to P_2, then we say that P_2 is a **refinement** of P_1. It is not difficult to show that in this case

$$L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1);$$

adding more points to a partition increases the lower sum and decreases the upper sum. (See Exercise 18 at the end of this section.) Given any two partitions, P_1 and P_2, we can form their **common refinement** P, which consists of all of the points of P_1 and P_2. Thus,

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2).$$

$U_n \geq U_{n+1}$



Hence, every lower sum is less than or equal to every upper sum. Since the real numbers are complete, there must exist *at least one* real number I such that

$$L(f, P) \leq I \leq U(f, P) \qquad \text{for every partition } P.$$

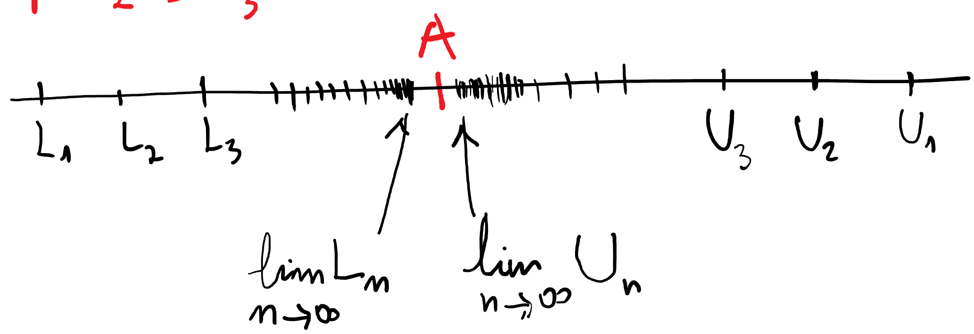
If there is *only one* such number, we will call it the definite integral of f on $[a, b]$.

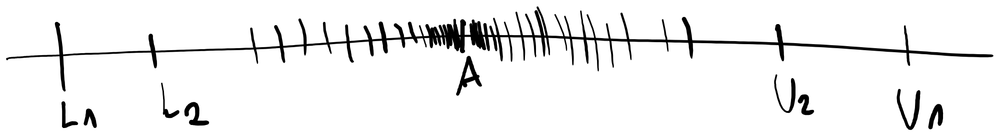
$L_1 \leq L_2 \leq L_3 \leq \cdots \leq A$

$U_1 \geq U_2 \geq U_3 \geq \cdots \geq A$

$L_1 \leq L_2 \leq L_3 \cdots \leq \underset{A}{A} \leq \cdots \leq U_3 \leq U_2 \leq U_1$

$$L_1 \le L_2 \le L_3 \dots \le A \le \dots \le U_3 \le U_2 \le U_1$$

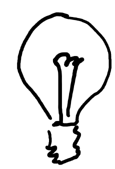




Om $\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$ då säger vi

att $\int_a^b f(x)\,dx$ finns och skriver

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n$$



När kommer detta att vara sant?

THEOREM 2

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

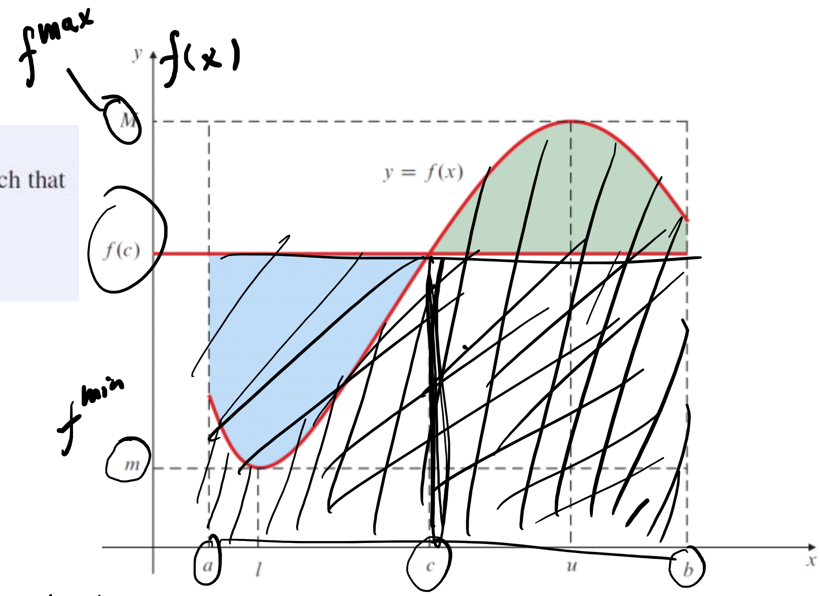
mellanvärdesatsen för integraller

den 30 september 2020 16:20

The Mean-Value Theorem for integrals

If f is continuous on $[a,b]$, then there exists a point c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$



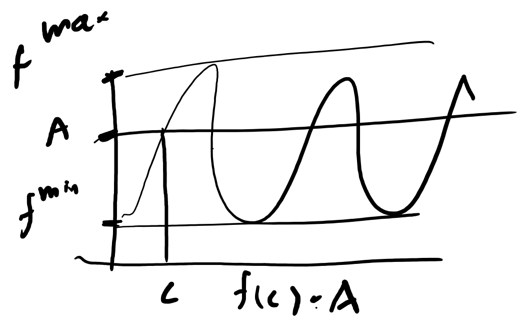
Bevis

$$L_1 \le L_2 \le \dots \int_a^b f\,dx \le \dots \le U_2 \le U_1$$

$$f^{min}(b-a) = L_1 \le \int_a^b f(x)\,dx \le U_1 = f^{max}(b-a) \quad \Big| :(b-a) \qquad b>a$$

$$f^{min} \le \overbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}^{A} \le f^{max}$$

mellan v.s.



$$\exists\, c \in [a,b] \text{ s.a } f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = (b-a)\,f(c)$$

den fundamentala satsen av analys

den 30 september 2020 16:33

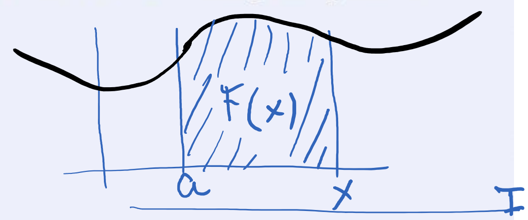
A(x)

The Fundamental Theorem of Calculus

Suppose that the function f is continuous on an interval I containing the point a.

PART I. Let the function F be defined on I by

$$F(x) = \int_a^x f(t)\,dt.$$

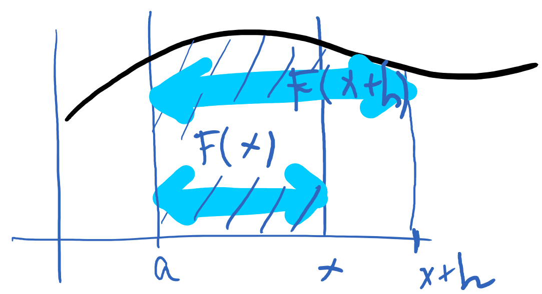


Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:

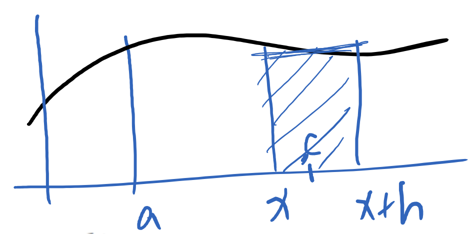
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$\cancel{\int}\cancel{d}f = f$ $\qquad$ $\cancel{\frac{d}{dx}}\cancel{\int^x} f(t)\,dt = f(x)$

PROOF Using the definition of the derivative, we calculate



$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)\\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad \text{by Theorem 3(d)}\\
&= \lim_{h\to 0}\frac{1}{\cancel{h}}\cancel{h}f(c) \qquad \text{for some } c = c(h) \text{ (depending on } h\text{)}\\
&\qquad\qquad \text{between } x \text{ and } x+h \text{ (Theorem 4)}\\
&= \lim_{c\to x} f(c) \qquad \text{since } c \to x \text{ as } h \to 0\\
&= f(x) \qquad \text{since } f \text{ is continuous.}
\end{aligned}$$



Q.E.D

PART II. If $G(x)$ is *any* antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I, we have

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

Bevis:

Om $F'(x) = f(x)$ då är det också sant att

$$(F(x) + c)' = f(x)$$

detta betyder att

$$\int_a^x f(t)\,dt = F(x) + c \qquad \int_0^x x^2\,dx = \frac{x^3}{3} + c$$

vi justerar konstanten c så att

$$\left.\begin{array}{c} \int_a^a f(t)\,dt = 0 \\ \| \\ F(a) + c = 0 \end{array}\right\} \Rightarrow \quad c = -F(a)$$

detta betyder att

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

vilket brukar skrivas i form

$$\int_a^b f(x)\,dx = F(b) - F(a) \overset{\text{def}}{=} F(x)\Big|_a^b$$

exempel hur man anväder AFS: ytan under x^2

den 1 oktober 2020 10:55

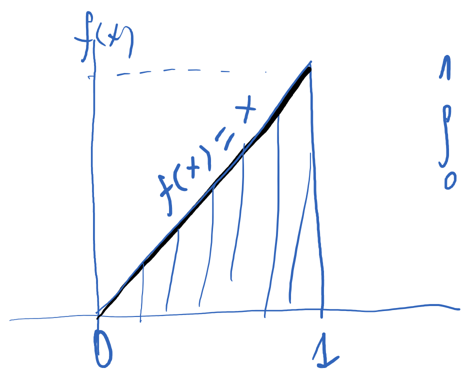
FSA = Fundamental Sats av Analys

AFS = Analysensens Fundamentalsats

$F'(x) = x^1$ $\quad \frac{d}{dx}(x^r) = r x^{r-1}$ $\quad r=2$ $\quad (x^2)' = 2x \mid :2$

$\left(\frac{x^2}{2}\right)' = x$

$\left(\frac{x^2}{2} + c\right)' = x$



$$\int_0^1 x\,dx = \frac{1}{2}\boxed{}1 = \frac{1}{2}1\cdot 1 = \frac{1}{2}$$

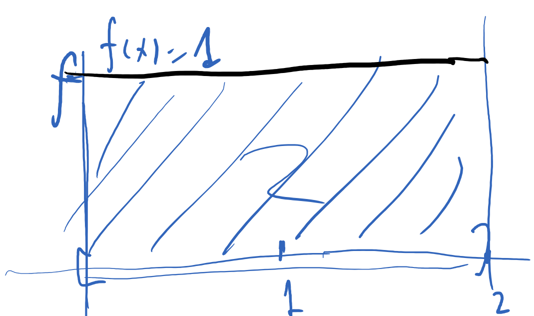
$$= F(x)\Big|_0^1 = F(1) - F(0)$$

$$= \left(\frac{x^2}{2} + c\right)\Big|_0^1$$

$$= \left(\frac{1^2}{2} + c\right) - \left(\frac{0^2}{2} + c\right)$$

$$= \frac{1^2}{2} - \frac{0^2}{2}$$

$$\int_0^1 x\,dx = \ldots = \frac{x^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$



$F'(x) = 1 \quad x' = 1$

$$\int_0^2 f(x)\,dx = \int_0^2 1\cdot dx = 2 \times 1 = 2$$

$$\overset{AFS}{=} x\Big|_0^2 = 2 - 0 = 2$$

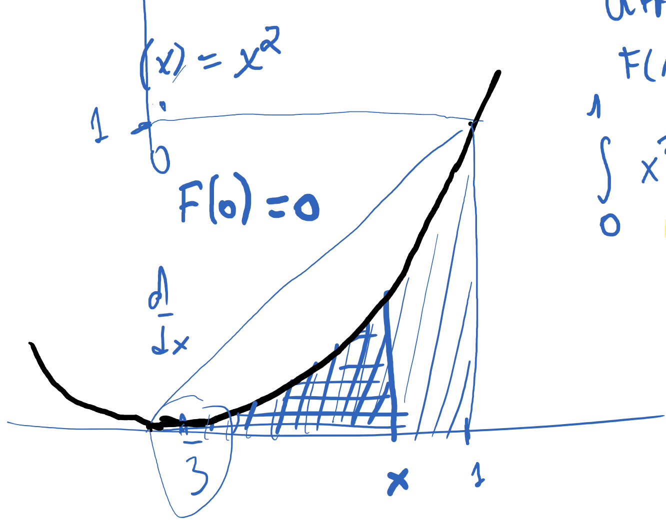
$c = 0$

$0 = F(0) = \frac{1}{3}0^3 + c$

Uträkning nedåt $\quad F(x) = \frac{1}{3}x^3 + c$

$F(1) = \frac{1}{3}1^3 + c = \frac{1}{3} + c = \frac{1}{3}$

$$\int_0^1 x^2\,dx = F(1) = \frac{1}{3} \qquad \frac{dF(x)}{dx} = f(x) = x^2$$



differential equation!

$F'(x) = x^2 \qquad F(x) = ?$

$\frac{d}{dx}x^r = r\cdot x^{r-1} \qquad r-1 = 2$

$r = 3$

$\frac{}{3}$ x 1

$$\frac{d}{dx} x^r = r \cdot x^{r-1}$$

$r-1=2$

$r=3$

$$(x^3) = 3x^2 \quad | : 3$$

$$\frac{d}{dx}(x^3) = x^2 \qquad \frac{dF}{dx} = x^2$$

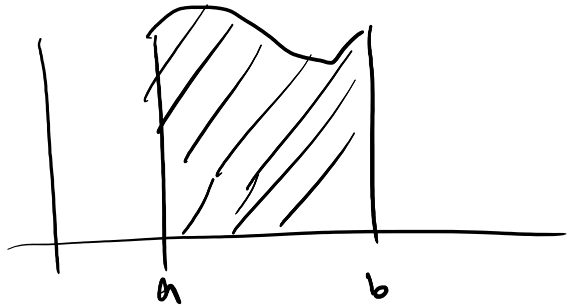
$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2 \qquad F(x) = \frac{1}{3}x^3 + c$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2$$

strategi att räkna ytor under grafer

den 30 september 2020 16:44

Problemet är att hitta ytan under grafen



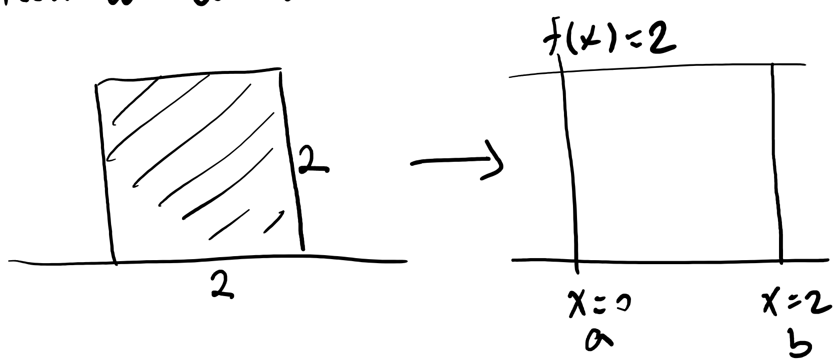
(1) hitta den primitiva funktionen, hitta $F(x)$ sådan att

$$F'(x) = f(x)$$ differential ekvation

(2) ytan är $F(b) - F(a)$

Exempel:

* ytan av en kvadrat



$$\int_0^2 2 \cdot dx = ?$$

$$f(x) = 2$$

$$F'(x) = ?$$

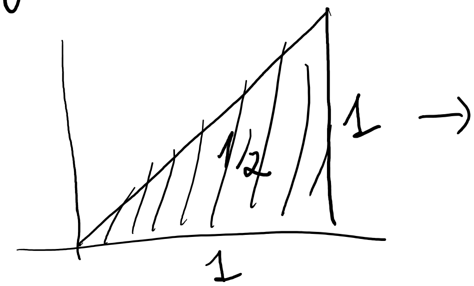
$$F(x) = 2x + c$$

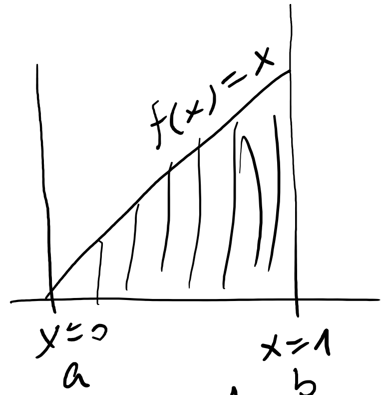
$$\int_0^2 f(x)\,dx = F(x)\Big|_0^2 = (2x+c)\Big|_0^2 = 2\cdot 2 + \cancel{c} - (\cancel{2}\cdot 0 + \cancel{c})$$

$$= 4$$

* ytan av en triangel

☆ ytan av en triangel





$$\int_0^1 x\,dx = ?$$

$$f(x) = x$$

$$F'(x) = f(x) = x$$

$$F(x) = \frac{1}{2}x^2 + c$$

$$\int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}$$

intuition/känsla

den 1 oktober 2020 11:32

$$\int_a^{x+dx} f(\tilde{x})\,d\tilde{x} = \int_a^{x} f(\tilde{x})\,d\tilde{x} + \int_x^{x+dx} f(\tilde{x})\,d\tilde{x}$$

$$\int f(\tilde{x})\,d\tilde{x} = \sum_{\tilde{x}} f(\tilde{x})\,d\tilde{x}$$

$$d\tilde{x} = dx$$

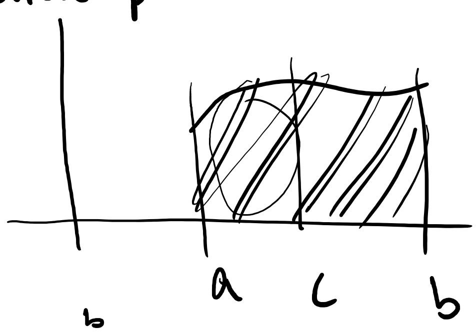
$$\int_a^{x+dx} f(\tilde{x})\,d\tilde{x} - \int_a^{x} f(\tilde{x})\,d\tilde{x} = \int_x^{x+dx} f(\tilde{x})\,d\tilde{x} \approx f(x)\cdot dx \qquad \Big| : dx$$

$$F(x) = \int^{x} f(\tilde{x})\,d\tilde{x}$$

$$\frac{\int_a^{x+dx} f\,d\tilde{x} - \int_a^{x} f\,d\tilde{x}}{dx} = f(x)$$

$$\frac{dF}{dx} = \frac{F(x+dx) - F(x)}{dx} = f(x)$$

Egenskaper



$$(1)\quad \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

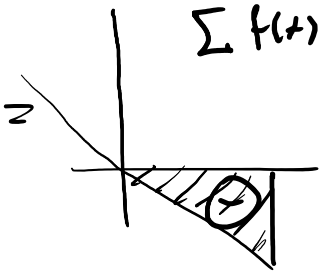
$$(2)\quad \int_a^b \left(f(x) + g(x) \right) dx = \underbrace{\int_a^b f(x)\,dx}_{F_1} + \underbrace{\int_a^b g(x)\,dx}_{F_2} \qquad F' = f + g$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{F_1' + F_2'}$$

$$(3)\quad \int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx \qquad (kF)' = kF' \qquad (F_1 + F_2)' \overset{\shortparallel}{}$$

$$\sum f(x)\,dx$$

$$\int_0^1 (-x)\,dx =$$



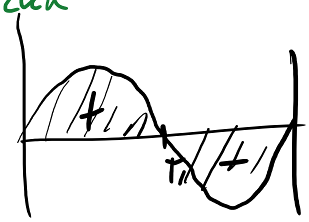
$$= -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$$

ka man inte ta absolutbeloppet på det eftersom en area inte kan vara negativ? → SVAR: nej, ta inte |..| om vi räknar den matematiska ytan ↓ men: ja om vi räknar den fysiska ytan

Vad händer då vid integralen av (1/x) från -1 till 1? Blir den 0 eller odefinierad?

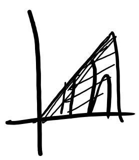
nästa vecka

$$\int_0^{2\pi} \sin(x)\,dx =$$



$$= 0$$

$$\int_0^1 x\,dx =$$



$$= \frac{1}{2}$$

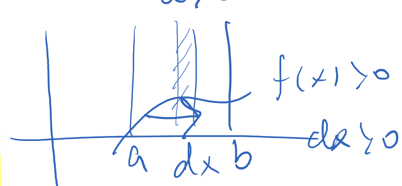
$$\int_1^0 x\,dx = \frac{x^2}{2}\Big|_1^0 = \frac{0^2}{2} - \frac{1^2}{2} = -\frac{1}{2} < 0$$

det här är konstig, varför är det så?

integral kan bli negativ trots att funktionen som integreras är positiv

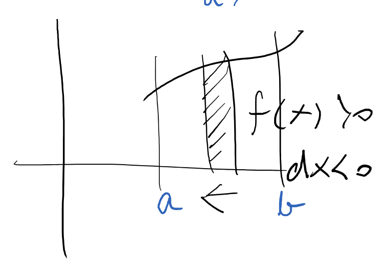
$$\sum \overset{<0}{\underset{>0}{f(x)}}\,\overset{>0}{\underset{<0}{dx}}$$

a<b



här är $\int_a^b f(x)\,dx > 0$

a>b



här är $\int_b^a f(x)\,dx < 0$

Kan man generalisera det och säga att integralen från a till b är samma som den negativa integralen från b till a?

JA: a<b

JA: $a > b$

(4) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$dx > 0$ $\quad$ $dx < 0$