integrations tekniker: intro

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att räkna ett integral ⟷ att lösa en differential ekvation

$$\int_a^b f(x)\,dx$$

$$F'(x) = f(x)$$

med randvillkor

$$F(a) = 0$$

I princip, man räknar integraler (löser differentiella ekvationer) med att gissa, för det mesta.

- gissa (utan vägledning)
- om man kan räkna en del av integral använd partiellintegration
- substitutionstekniker

Using Integral Tables

You can get some help evaluating integrals by using an integral table, such as the one in the back endpapers of this book. Besides giving the values of the common elementary integrals that you likely remember while you are studying calculus, they also give many more complicated integrals, especially ones representing standard types that often arise in applications. Familiarize yourself with the main headings under which the integrals are classified. Using the tables usually means massaging your integral using simple substitutions until you get it into the form of one of the integrals in the table.

Lärande mål!

Det går också att räkna integraler numeriskt:

- trapez regel
- Simpson regel
- feluppskattning $\mathcal{O}(h), \mathcal{O}(h^2), \ldots$

sämre approximation

bättre approximation

integral typer

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5.5 The Fundamental Theorem of Calculus

In this section we demonstrate the relationship between the definite integral defined in Section 5.3 and the indefinite integral (or general antiderivative) introduced in Section 2.10. A consequence of this relationship is that we will be able to calculate definite integrals of functions whose antiderivatives we can find.

DEFINITION 3

The definite integral

Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have

$$L(f, P) \le I \le U(f, P).$$

Then we say that the function f is **integrable** on $[a, b]$, and we call I the **definite integral** of f on $[a, b]$. The definite integral is denoted by the symbol

$$I = \int_a^b f(x)\,dx.$$

DEFINITION 7

An **antiderivative** of a function f on an interval I is another function F satisfying

$$F'(x) = f(x) \quad \text{for } x \text{ in } I.$$

2.10 Antiderivatives and Initial-Value Problems

DEFINITION 8

The **indefinite integral** of $f(x)$ on interval I is

$$\int f(x)\,dx = F(x) + C \qquad \text{on } I,$$

provided $F'(x) = f(x)$ for all x in I.

Det finns två typer av integraller:

- bestämda tex. $\int_1^2 x^2 dx = \text{tal}$
- obestämda tex $\int x^2 dx = \text{algebraisk uttryck i } x$ $= \frac{x^3}{3} + C$

6.5 Improper Integrals

Improper integrals of type I

If f is continuous on $[a, \infty)$, we define the improper integral of f over $[a, \infty)$ as a limit of proper integrals:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Similarly, if f is continuous on $(-\infty, b]$, then we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (is a finite number), we say that the improper integral **converges**; if the limit does not exist, we say that the improper integral **diverges**. If the limit is ∞ (or $-\infty$), we say the improper integral **diverges to infinity** (or **diverges to negative infinity**).

Improper integrals of type II

If f is continuous on the interval $(a, b]$ and is possibly unbounded near a, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx.$$

EXAMPLE 5 Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

Similarly, if f is continuous on $[a, b)$ and is possibly unbounded near b, we define

$$\int_a^b f(x)\,dx = \lim_{c\to b-} \int_a^c f(x)\,dx.$$

These improper integrals may converge, diverge, diverge to infinity, or diverge to negative infinity.

Beware of integrals of the form $\int_a^b f(x)\,dx$ where f is not continuous at *all* points in the interval $[a, b]$. The Fundamental Theorem does not apply in such cases.

EXAMPLE 6 We know that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ if $x \neq 0$. It is *incorrect*, however, to state that

$$\int_{-1}^{1} \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0 - 0 = 0,$$

even though $1/x$ is an odd function. In fact, $1/x$ is undefined and has no limit at $x = 0$, and it is not integrable on $[-1, 0]$ or $[0, 1]$ (Figure 5.25). Observe that

$$\lim_{c\to 0+} \int_{c}^{1} \frac{1}{x}\,dx = \lim_{c\to 0+} -\ln c = \infty,$$

so both shaded regions in Figure 5.25 have infinite area. Integrals of this type are called **improper integrals**. We deal with them in Section 6.5.

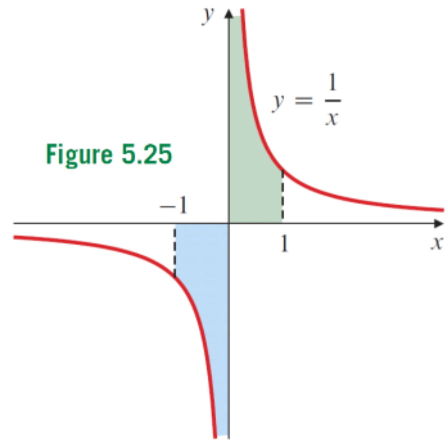

Figure 5.25

att gisa: använd tabeller med derivator

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$$\text{funktion} \xleftarrow{\int} \text{derivata}$$

$$\text{funktion} \xrightarrow{d/dx} \text{derivata}$$

$f(x)$	$f'(x)$
x^r	$r x^{r-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan x$	$\frac{1}{\cos^2 x}$
e^x	e^x
$\ln x$	$\frac{1}{x}$

$$\int f'(x)\,dx = f(x) + c$$

$$\int^x f(t)\,dt = F(x), \quad F'(x) = f(x)$$

$$\int^x F'(t)\,dt = F(x) + c \quad F'(x) = f(x)$$

$$\int F'(x)\,dx = F(x) + c$$

__exempel av användning av tabellen__

$$(x^r)' = r x^{r-1} \quad \Big| \int VS\,dx = \int HS\,dx$$

$$\int r x^{r-1}\,dx = \int (x^r)'\,dx$$

$$\int r x^{r-1}\,dx = x^r + c$$

städa konstanten

$$r \int x^{r-1}\,dx = x^r + c$$

dela med r

$$\int x^{r-1}\,dx = \frac{x^r}{r} + \frac{c}{r}$$

byt $r \to r+1$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + \frac{c}{r+1}$$

Om c är en konstant så är det också $c/(r+1) = \tilde{c}$

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + \tilde{c}$$

exempel 2:

$(\sin x)' = \cos x \quad \Big| \int VS\,dx = \int HS\,dx$

$\int (\sin x)'\,dx = \int \cos x\,dx$

$\|$ integral och derivata är "invers" operationer till varandra

$\sin x + c$

$\int \cos x\,dx = \sin x + c$

exempel 3:

$(\cos x)' = -\sin x \quad \Big| \int$

$\int (\cos x)'\,dx = \int (-\sin x)\,dx$

$\cos x + c = -\int \sin x\,dx \quad \Big| (-1)$

$-\cos x - c = \int \sin x\,dx$

döpa om konstanten

$-\cos x + \tilde{c} = \int \sin x\,dx$

Exempel 3:

5. Calculate the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$. Hence, express each of the indefinite integrals

$$\int \frac{dx}{\sqrt{x^2+1}}, \qquad \int \frac{dx}{\sqrt{x^2-1}}, \qquad \int \frac{dx}{1-x^2}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \int \left(\sinh^{-1}(x)\right)'\,dx = \sinh^{-1}(x) + c$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \int \left(\cosh^{-1}(x)\right)'\,dx = \cosh^{-1}(x) + c$$

$$\int \frac{\quad}{\sqrt{x^2-1}} = \int (\cosh\ (x))\ dx = \cosh\ (x) + c$$

$$\int \frac{dx}{1-x^2} = \int \left(\tanh^{-1}(x)\right)' dx = \tanh^{-1}(x) + c$$

variabel byte

den 1 oktober 2020 15:19

5.6 The Method of Substitution

6.3 Inverse Substitutions

Example:

$$\int (x+1)^2\,dx = \int (x^2+2x+1)\,dx$$

$$= \int x^2\,dx + 2\int x\,dx + \int dx$$

$$= \frac{x^3}{3} + 2\,\frac{x^2}{2} + x + c$$

$$= \frac{x^3}{3} + x^2 + x + c$$

$$\int (x+1)^2\,dx = \{\, u = x+1 \,,\ du = dx \,\} = \int u^2\,du = \frac{u^3}{3} + c$$

$$= \frac{1}{3}(x+1)^3 + c = \frac{1}{3}\left(x^3 + 3x^2 + 3x + 1\right) + c$$

$$= \frac{1}{3}x^3 + x^2 + x + \frac{1}{3} + c$$

$$= \frac{1}{3}x^3 + x^2 + x + \tilde{C}$$

det var kanske inte nödvändig att använda variabel byte här men det blir nödvändig för nästa exempel:

$$\int (x+1)^{100}\,dx = \int \big(\underbrace{x^{100} + \dots + 1}_{\text{ganska många termer}}\big)\,dx$$

$$\int (x+1)^{100}\,dx = \{ u = x+1,\ du = dx \} = \int u^{100}\,du$$

$$= \frac{u^{101}}{101} + C = \frac{(x+1)^{101}}{101} + C$$

Ett exempel till:

$$\int \frac{dx}{4+x^2} = \{ x = 2u,\ dx = 2du \} = \int \frac{2\,du}{4+4u^2} = \frac{2}{4}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\int \frac{du}{1+u^2} \qquad (\arctan u)' = \frac{1}{1+u^2}$$

$$= \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

Det finns en grej att vara försiktig om när det gäller variabel byte med bestämda integraller.

Substitution in a definite integral

Suppose that g is a differentiable function on $[a, b]$ that satisfies $g(a) = A$ and $g(b) = B$. Also suppose that f is continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_A^B f(u)\,du.$$

THEOREM 6

$$\int_0^2 \frac{dx}{4+x^2} = \left\{ \begin{matrix} x = 2u & x=0 & u=0 \\ dx = 2du & x=2 & u=1 \end{matrix} \right\} = \int_0^1 \frac{2\,du}{4+4u^2}$$

$$= \frac{2}{4}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\arctan(u)\Big|^1$$

$$= \frac{2}{4}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\arctan(u)\Big|_0^1$$

$$= \frac{1}{2}\left[\arctan(1) - \arctan(0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$$

partiell integration

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6.1 Integration by Parts

$$\frac{d}{dx}\left(U(x)V(x)\right) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx} \qquad \Big| \int VS\,dx = \int HS\,dx$$

$$\int \frac{d}{dx}(UV)\,dx = \int U\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx$$

$$UV = \int U\frac{dV}{dx}\,dx + \int V\frac{dU}{dx}\,dx$$

$$\int U(x)\frac{dV}{dx}\,dx = U(x)V(x) - \int V(x)\frac{dU}{dx}\,dx$$

Kanske enklare att tänka så här:

$$d(uv) = du\,v + u\,dv \qquad \Big| \int$$

$$\int d(uv) = \int du\,v + \int u\,dv$$

$$uv = \int du\,v + \int u\,dv$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example: när man kan räkna en del av integrallen;

$$\int x\sin x\,dx = \left\{ u = x,\ du = dx \quad \ldots \right\} =$$

$$\int \underset{u}{x} \underbrace{\sin x \, dx}_{dv} = \left\{ \begin{array}{l} u = x, \quad du = dx \\ dv = \sin x \, dx, \quad v = \int \sin x \, dx = -\cos x \end{array} \right\} =$$

$$= uv - \int v \, du = -x \cos x - \int (-\cos x) \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + c$$

testa: $(-x \cos x + \sin x + c)' = -(x \cos x)' + (\sin x)' + c'$

$$= -(1 \cos x + x(-\sin x)) + \cos x + 0$$

$$= -\cos x + x \sin x + \cos x = x \sin x$$

numerisk integration

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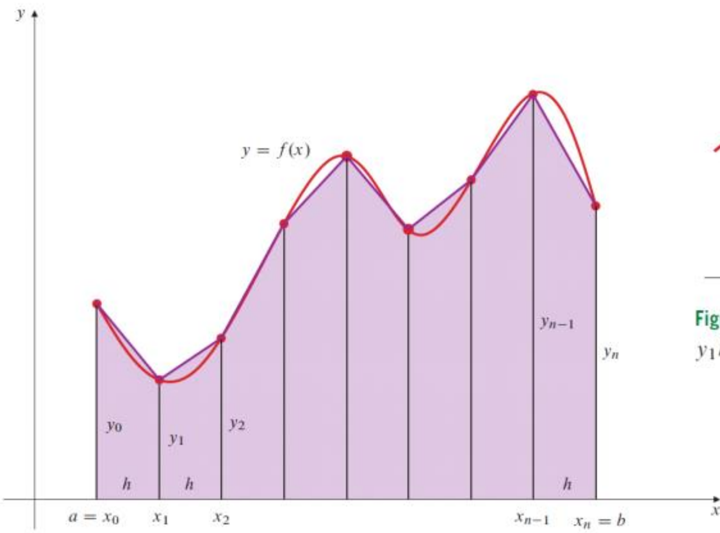
6.6 The Trapezoid and Midpoint Rules



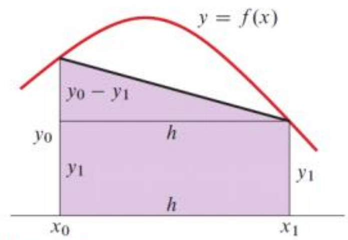
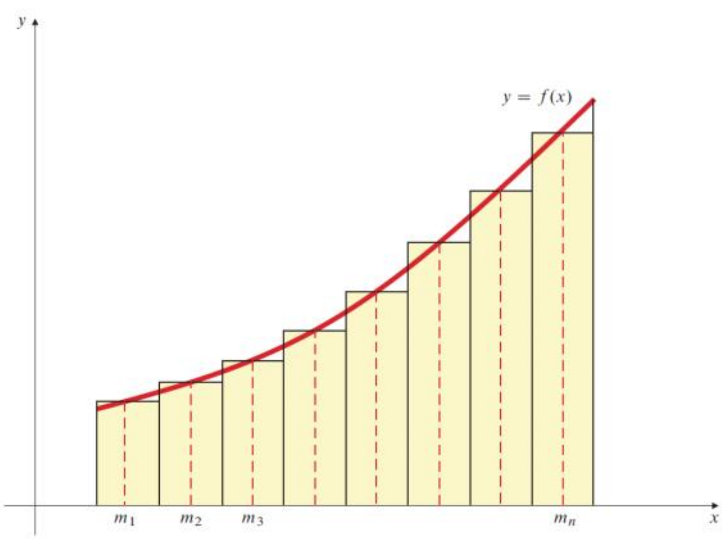


Figure 6.16 The trapezoid has area $y_1 h + \frac{1}{2}(y_0 - y_1)h = \frac{1}{2}h(y_0 + y_1)$

$$\int_a^b f(x)\,dx \approx h\left(\frac{y_0+y_1}{2} + \frac{y_1+y_2}{2} + \frac{y_2+y_3}{2} + \cdots + \frac{y_{n-1}+y_n}{2}\right)$$
$$= h\left(\frac{1}{2}y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{1}{2}y_n\right).$$

The Midpoint Rule



Error estimates for the Trapezoid and Midpoint Rules

If f has a continuous second derivative on $[a, b]$ and satisfies $|f''(x)| \le K$ there, then

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{K(b-a)}{12}h^2 = \frac{K(b-a)^3}{12n^2},$$

$$\left|\int_a^b f(x)\,dx - M_n\right| \le \frac{K(b-a)}{24}h^2 = \frac{K(b-a)^3}{24n^2},$$

where $h = (b-a)/n$. Note that these error bounds decrease like the square of the subinterval length as n increases.

6.7 Simpson's Rule

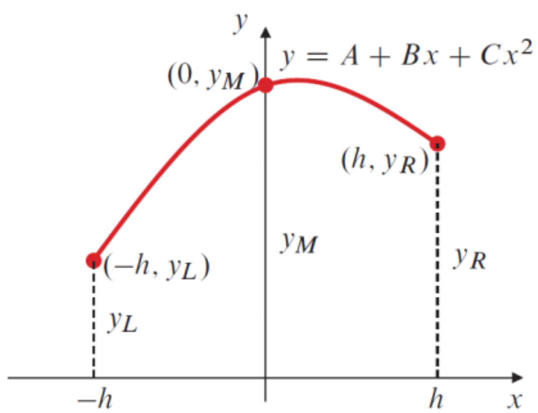


Figure 6.23 Fitting a quadratic graph through three points with equal horizontal spacing

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx \approx \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n).$$

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

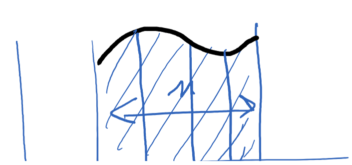
$$= \frac{h}{3}\left(\sum y_{\text{"ends"}} + 4\sum y_{\text{"odds"}} + 2\sum y_{\text{"evens"}}\right).$$

Error estimate for Simpson's Rule

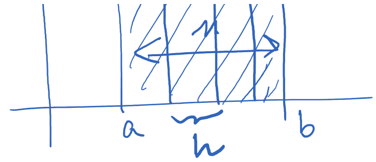
If f has a continuous fourth derivative on the interval $[a, b]$, satisfying $|f^{(4)}(x)| \le K$ there, then

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{K(b-a)}{180}h^4 = \frac{K(b-a)^5}{180n^4},$$

where $h = (b-a)/n$.



where $h = (b-a)/n$.



$$b - a = n \cdot h$$

$$\left| \int_a^b f(x) - T_n \right| \le \frac{K(b-a)}{12} h^2 \quad \overset{|f''(x)| \le K}{\Longrightarrow} \quad \int_a^b f(x)\,dx = T_n + O(h^2)$$

$$\left| \int_a^b f(x) - M_n \right| \le \frac{K(b-a)}{24} h^2 \quad \overset{|f''(x)| \le K}{\Longrightarrow} \quad \int_a^b f(x)\,dx = M_n + O(h^2)$$

$$\left| \int_a^b f(x) - S_n \right| \le \frac{K(b-a)}{180} h^4 \quad \overset{|f^{(4)}(x)| \le K}{\Longrightarrow} \quad \int_a^b f(x)\,dx = S_n + O(h^4)$$