differentiala ekvationer: intro

den 12 oktober 2020 11:54

~~typer~~
~~måååånga exempel~~
~~relation till integraler~~
som modelleringsverktyg
roll som den fria konstanten spelar
system av differentiella ekvationer

DE som modellerings verktyg

den 12 oktober 2020 13:05

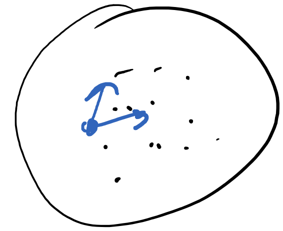
Exponential Growth and Decay Models

Many natural processes involve quantities that increase or decrease at a rate proportional to their size. For example, the mass of a culture of bacteria growing in a medium supplying adequate nourishment will increase at a rate proportional to that mass. The value of an investment bearing interest that is continuously compounding increases at a rate proportional to that value. The mass of undecayed radioactive material in a sample decreases at a rate proportional to that mass.

All of these phenomena, and others exhibiting similar behaviour, can be modelled mathematically in the same way. If $y = y(t)$ denotes the value of a quantity y at time t, and if y changes at a rate proportional to its size, then

$$\frac{dy}{dt} = ky.$$

$y(t)$ mängd



$$\frac{dy(t)}{dt} = k\cdot y(t)$$

Sometimes an exponential growth or decay problem will involve a quantity that changes at a rate proportional to the difference between itself and a fixed value:

$$\boxed{\frac{dy}{dt} = k(y - a).}$$

gymnasie fysik

$$F = a\cdot m$$

$$\frac{F}{m} = a = \frac{dv}{dt}, \quad \frac{dx}{dt} = v$$

$$\frac{F(t)}{m} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$F(t) = 0$$

$$\boxed{0 = \frac{d^2x}{dt^2} = x''(t)}$$

$$0 = y'(t)$$

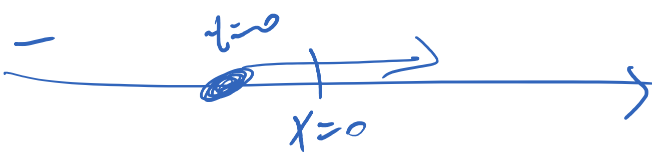
$$y(t) = \text{konst}$$

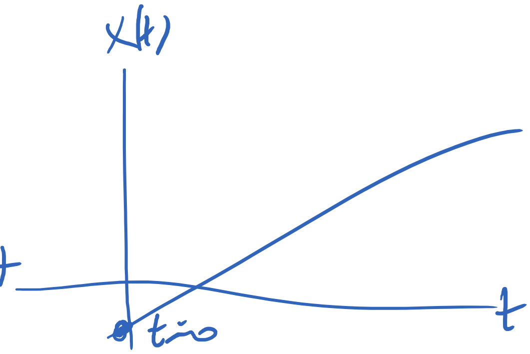
$v = x'(t)$

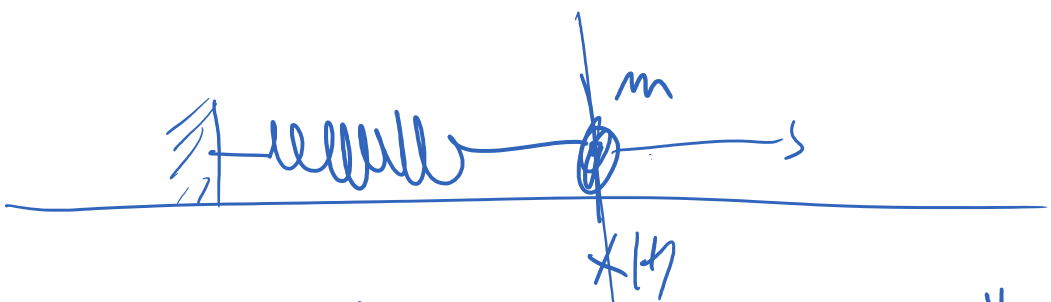
$v_0 = x'(t)$

$v_0 t + x_0 = x(t)$

$y(t) = \text{konst}$







$$F = -k \cdot (x - x_*) = m \cdot x''(t)$$

$$\boxed{-k\,(x(t) - X_*) = m \cdot x''(t)}$$

model

$x(t)$: $\sin(\omega t)$

$\cos(\omega t)$

typer av differentiella ekvationer

den 12 oktober 2020 12:29

$\boxed{\text{AD/ 18.1 Classifying DE}}$

- ordinär differential ekvation ODE

$$\frac{df}{dx} = g(f, x) \quad f(x)$$

$$\frac{df}{dx} = -f^2 + 1 + \cos x$$

- partiell dif. ekvation $f(x)$, $u(x,t)$

$$\frac{\partial^2}{\partial t^2} u(x,t) = c \frac{\partial^2}{\partial x^2} u(x,t)$$

$$\boxed{\underline{A} \cdot \underline{x} = \underline{a}}$$

- linjär ekvation | homogen

$$y''(x) + k y(x) = 0$$

$$\boxed{D y(x) = 0}$$

operator

$$\cancel{y''(x) + k y^2(x) = 0}$$

$$D y(x) = \cos x$$

$$\cancel{\sin y = y'}$$

icke homogen

$$\boxed{D = \frac{d^2}{dx^2} + k \mathbb{1}}$$

$$D y = \left(\frac{d^2}{dx^2} + k \mathbb{1}\right) y = \frac{d^2}{dx^2} y + k y$$

identitet " $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ "

identitet $\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix} = \mathbb{I}$

$$\mathcal{D} = \frac{d^2}{dx^2} + k \qquad \mathcal{D}y = \frac{d^2y}{dx^2} + k\cdot y$$

★ Den allmäna formen för linjär DE :

$$a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \dots + a_2(x)\, y''(x) +$$

$$+ a_1(x)\, y'(x) + a_0(x)\, y(x) = f(x)$$

koefficient funktioner

y'', y', y, x

$$x^2 y''(x) + x\, y'(x) + y(x) = \ln x$$

linjär

icke homogen

grad 2

icke konst. koefficienter

ODE

separabla ODE

den 12 oktober 2020 12:31

18.2 Solving First-Order Equations

7.9 First-Order Differential Equations

→ ja av grad 1
annars inte

Är alla linjära DE separabla?

icke separabel

$\frac{dy}{dx} = \sin(x \cdot y)$

* separations teknik

$$\boxed{\frac{dy}{dx} = f(x)g(y)} \quad \Big| : g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

Separable Equations

Consider the logistic equation introduced in Section 3.4 to model the growth of an animal population with a limited food supply:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right), \quad \Big| \cdot dt$$

where $y(t)$ is the size of the population at time t, k is a positive constant related to the fertility of the population, and L is the steady-state population size that can be sustained by the available food supply. This equation has two particular solutions, $y = 0$ and $y = L$, that are constant functions of time.

* exempel

$$\frac{dy}{dx} = x \cdot y \quad \Big| : y$$

$$\frac{1}{y}\frac{dy}{dx} = x \quad \Big| \cdot dx$$

$$\frac{1}{y}\,dy = x\,dx \quad \Big| \int$$

$$\int \frac{dy}{y} = \int x\,dx$$

$$\ln|y| + C_1 = \frac{x^2}{2} + C_2$$

C_3

$$\int \frac{L\,dy}{y(L-y)} = \int k\,dt \Rightarrow \int \left(\frac{1}{y} + \frac{1}{L-y}\right) dy = kt + C.$$

Assuming that $0 < y < L$, we therefore obtain

$$\ln y - \ln(L-y) = kt + C,$$

$\ln|y| - \ln|L-y|$

$$\exp\left(\ln\left(\frac{y}{L-y}\right)\right) = kt + C. \quad \Big| \exp$$

$$\frac{y}{L-y} = e^{kt+C} = C_1 e^{kt} \qquad C_1 = e^C$$

$$y = (L-y)C_1 e^{kt}$$

$$\boxed{y = \frac{C_1 L e^{kt}}{1 + C_1 e^{kt}},}$$

$y(0) = ?$

$$\ln|y| = \frac{x^2}{2} + \overbrace{c_2 - c_1}^{c_3}$$

$$\ln|y| = \frac{x^2}{2} + c_3 \quad \Big| \; \exp$$

$$|y| = e^{\frac{x^2}{2}} e^{c_3}$$

Speciell fall:

$$\frac{dy}{dx} = f(x) \cdot \cancelto{1}{g(y)}$$

$$\frac{dy}{dx} = f(x) \quad \Big| \; \int_{x_1}^{x_2} \ldots \, dx$$

$$\underbrace{\int_{x_1}^{x_2} \frac{dy}{dx} \, dx} = \int_{x_1}^{x_2} f(x) \, dx$$

$$y(x_2) - y(x_1) = \int_{x_1}^{x_2} f(x) \, dx$$

$\cancel{x_2}$ x

x_1

$$y(\cancel{x_2}\,x) = y(\cancel{x_1}\,x_0) + \int_{\cancel{x_1}\,x_0}^{\cancel{x_2}\,x} f(\cancel{x}\,u)\,d\cancel{x}\,u$$

$$y(x) = \underbrace{y(x_0)}_{\text{begynnelse villkor}} + \boxed{\int_{x_0}^{x} f(u)\,du} \qquad \cancel{\int_{x_0}^{x} f(x)\,dx}$$

begynnelse villkor

$y(x_0) = \text{tal}$

1:a grad ODE

den 12 oktober 2020 12:30

$\boxed{\text{AD/7.9 1:a grad ODE}}$

7.9 First-Order Differential Equations

Vrf var den icke linjär?

First-Order Linear Equations

A first-order **linear** differential equation is one of the type

$$\frac{dy}{dx} + p(x)y = q(x), \qquad (*)$$

where $p(x)$ and $q(x)$ are given functions, which we assume to be continuous. The equation is called **nonhomogeneous** unless $q(x)$ is identically zero. The corresponding **homogeneous** equation,

$$\frac{dy}{dx} + p(x)y = 0,$$

$q(x)$

is separable and so is easily solved to give $y = K\,e^{-\mu(x)}$, where K is any constant and $\mu(x)$ is any antiderivative of $p(x)$:

$$\mu(x) = \int p(x)\,dx \quad \text{and} \quad \frac{d\mu}{dx} = p(x).$$

$$a(x)y'(x) + b(x)y(x) = c(x) \quad \Big|\, :a(x)$$

$$y'(x) + \boxed{\frac{b(x)}{a(x)}}\,y(x) = \boxed{\frac{c(x)}{a(x)}}$$

$$p(x) \qquad q(x)$$

$$\frac{dy}{dx} = -p(x)y$$

$$\int \frac{dy}{y} = -\int p\,dx$$

$$\mu$$

$$\ln y = -\int p(x)\,dx = -\mu(x)$$

$$y = e^{-\mu(x)}$$

METHOD I **Using an Integrating Factor.** Multiply equation $(*)$ by $e^{\mu(x)}$ (which is called an **integrating factor** for the equation) and observe that the left side is just the derivative of $e^{\mu(x)}y$; by the Product Rule

P.r.

$$\frac{d}{dx}\left(e^{\mu(x)}y(x)\right) = e^{\mu(x)}\frac{dy}{dx} + e^{\mu(x)}\frac{d\mu}{dx}y(x)$$

$$= e^{\mu(x)}\left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)}q(x).$$

$$(e^{\mu})' = e^{\mu}\mu'$$

$$\frac{d}{dx}\left(e^{\mu}y\right) = e^{\mu}q \qquad \int$$

Therefore, $e^{\mu(x)}y(x) = \int e^{\mu(x)}q(x)\,dx$, or

$$\boxed{y(x) = e^{-\mu(x)}\int e^{\mu(x)}q(x)\,dx}$$

$$e^{-\mu}\;\Big|\; e^{\mu}y = \int e^{\mu}q\,dx$$

$$y = e^{-\mu}\int e^{\mu}q\,dx$$

1:a grad ODE: exempel AD/7.9:ex 7

den 12 oktober 2020 13:00

EXAMPLE 7 Solve $\frac{dy}{dx} + \frac{y}{x} = 1$ for $x > 0$. Use both methods for comparison.

$$y' + \frac{1}{x} y = 1 \quad \leftrightarrow \quad y' + p \cdot y = q$$

$p = \frac{1}{x}$

$q = 1$

(1) $\mu = \int p \, dx$

(2) $y = e^{-\mu \, {}^{-c}} \int e^{\mu \, {}^{+c}} q \, dx$

(1) $\mu = \int \frac{1}{x} dx = \ln x + C_1$

så 1/e^kx integral e^kx = 1/x integral x?

$k = 1$

ja om $k = 2$

$e^{\ln x} = x$

(2) $y = e^{-(\ln x + C_1)} \int e^{+(\ln x + C_1)} 1 \cdot dx$

$e^{\ln(k \cdot x)} = k \cdot x$

$e^{-\ln(k x)} = \frac{1}{k \cdot x}$

$$= e^{-\ln x} \cancel{e^{-C_1}} \int e^{\ln x} \cancel{e^{C_1}} dx$$

$$= \frac{1}{e^{\ln x}} \int e^{\ln x} dx = \frac{1}{x} \int x \, dx$$

$$= \frac{1}{x} \left(\frac{x^2}{2} + c \right) = \frac{x^{\cancel{2}}}{2\cancel{x}} + \frac{c}{x}$$

$$y = \frac{x}{2} + \frac{c}{x} \quad \cancel{+ \ldots}$$

$y(0)$ $\boxed{y(1) = 3}$

$3 = \frac{1}{2} + \frac{c}{1}$

1:a grad linjära ekvationer med konstanta koefficienter

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den allmäna formen:

$$a(x)y'(x) + b(x)y(x) = c(x) \quad | : a(x)$$

$$y'(x) + \frac{b(x)}{a(x)}y(x) = \frac{c(x)}{a(x)}$$

$$\boxed{y'(x) + \overset{\lambda}{p(x)}y(x) = q(x)}$$

$$y' = ry$$

$$y'(x) + \lambda y(x) = \cancel{q(x)}\ 0$$

$$\frac{dy}{y} = r\,dx$$

ansatz: $y(x) = Ae^{rx}$

$$y'(x) = A(e^{rx})' = Ae^{rx}(r)$$

$$y = Ae^{r\cdot x} \qquad y' = rAe^{rx} \qquad y = 0$$

$$y' + \lambda y = rAe^{rx} + \lambda Ae^{rx} = \underbrace{(r+\lambda)}\,Ae^{r\cdot x} = 0$$

$$\neq 0$$

$$r + \lambda = 0 \qquad r = -\lambda$$

$y' + \lambda y = 1$

$y_h' + \lambda y_h = 0$ lös

$y_p' + \lambda y_p = 1$ gissa

$$\boxed{y = y_h + y_p}$$

$y = Ae^{-\lambda x} + \frac{1}{\lambda}$

$y(0) = ?$

$y_h = Ae^{-\lambda x}$

$\lambda y_p = 1 \qquad y_p = 1/\lambda$

$y' + \lambda y = x$

2:a grad linjära med konstanta koefficienter

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$\boxed{\text{AD/18.4 Second-order linear eqs.}}$

$$\underbrace{a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x)}_{\mathcal{D}f \text{ operator form}} = f(x)$$

$$\mathcal{D} = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\,\mathbb{1}$$

$\mathcal{D}\,y_h(x) = 0$ $\qquad y_h(x) = c_1 u_1(x) + c_2 u(x)$

$\mathcal{D}\,y_p(x) = f(x)$ $\qquad$ Linjäroberoende funktioner

$y(x) = y_h(x) + y_p(x)$ är lösningen till $\mathcal{D}y(x) = f(x)$

* andra grad DE med konstanta koefficienter

AD/18.5 Linear DE with Constant Coeffs.

AD/18.6 Nonhomogeneous linear Eqs.

$a\,y''(x) + b\,y'(x) + c\,y(x) = 0$ AD/18.5

$= f(x)$ AD/18.6

ansatz: $y_h(x) = Ae^{r\cdot x}$

[illegible]

ansatz: $y_h(x) = Ae^{rx}$

$a\, y_h''(x) + b y_h' + c y_h = 0$

$y = y_h + y_p$

$y_h^{(0)} = r^0 A e^{rx}$

$y_h' = r A e^{rx}$

$a r^2 A e^{rx} + b r A e^{rx} + c A e^{rx} = 0$

$y_h'' = r^2 A e^{rx}$

$\underbrace{(ar^2 + br + c)}_{0} A e^{rx} = 0$

$\cancel{A = 0}$ $\neq 0$

$r_{1,2} = \ldots$

$\mathcal{D} y = 0$

$\mathcal{D} y_h + \underbrace{\mathcal{D} A_1 e^{r_1 x}}_{0} + \underbrace{\mathcal{D} A_2 e^{r_2 x}}_{0} = 0$ $\mathcal{D}0 = 0$

DE grad 1 $\rightarrow$ 1 fri konst

2 $\rightarrow$ 2 fria konst

⋮