komplexa tal och differentiella ekvationer

den 13 oktober 2020 17:06

komplexa tal
- viktiga definitioner
- egenskaper
- operationer

komplex tal: polärform

den komplexa funktionen
- definition
- egenskaper
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relationen mellan trigonometriska och hyperboliska funktioner

användning av komplexa tal för att lösa differentiella ekvationer komplexa tal

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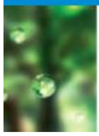
APPENDIX I



Complex Numbers

DEFINITION 1

A **complex number** is an expression of the form

$$a + bi \qquad \text{or} \qquad a + ib,$$

where a and b are *real numbers*, and i is the imaginary unit.

DEFINITION 2

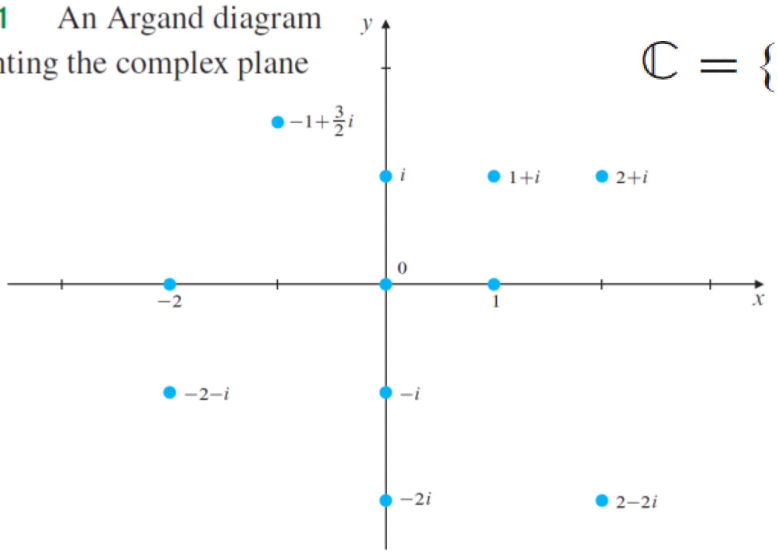
If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $\text{Re}\,(z)$. We call y the **imaginary part** of z and denote it $\text{Im}\,(z)$:

$$\text{Re}\,(z) = \text{Re}\,(x + yi) = x, \qquad \text{Im}\,(z) = \text{Im}\,(x + yi) = y.$$

$$\begin{aligned} &\text{Re}\,(3 - 5i) = 3 && \text{Im}\,(3 - 5i) = -5 \\ &\text{Re}\,(2i) = \text{Re}\,(0 + 2i) = 0 && \text{Im}\,(2i) = \text{Im}\,(0 + 2i) = 2 \\ &\text{Re}\,(-7) = \text{Re}\,(-7 + 0i) = -7 && \text{Im}\,(-7) = \text{Im}\,(-7 + 0i) = 0. \end{aligned}$$

Graphical Representation of Complex Numbers

Figure I.1 An Argand diagram representing the complex plane



$$\mathbb{C} = \{x + yi \ : \ x,\ y, \in \mathbb{R}\}$$

DEFINITION 3

The distance from the origin to the point (a, b) corresponding to the complex number $w = a + bi$ is called the **modulus** of w and is denoted by $|w|$ or $|a + bi|$:

$$|w| = |a + bi| = \sqrt{a^2 + b^2}.$$

DEFINITION 4

If the line from the origin to (a, b) makes angle θ with the positive direction of the real axis (with positive angles measured counterclockwise), then we call θ an **argument** of the complex number $w = a + bi$ and denote it by $\arg(w)$ or $\arg(a + bi)$. (See Figure I.2.)

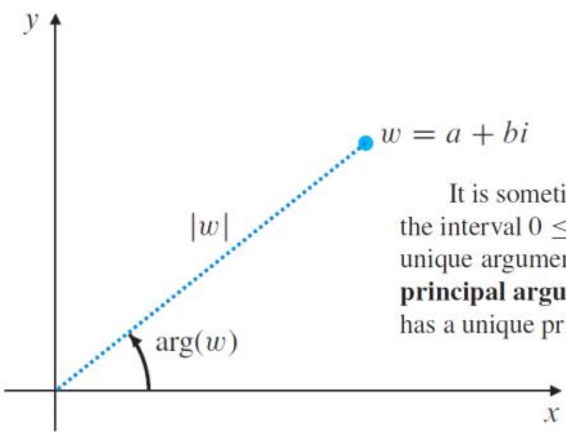


Figure I.2 The modulus and argument of a complex number

It is sometimes convenient to restrict $\theta = \arg(w)$ to an interval of length 2π, say, the interval $0 \leq \theta < 2\pi$, or $-\pi < \theta \leq \pi$, so that nonzero complex numbers will have unique arguments. We will call the value of $\arg(w)$ in the interval $-\pi < \theta \leq \pi$ the **principal argument** of w and denote it $\operatorname{Arg}(w)$. Every complex number w except 0 has a unique principal argument $\operatorname{Arg}(w)$.

Remark If $z = x + yi$ and $\operatorname{Re}(z) = x > 0$, then $\operatorname{Arg}(z) = \tan^{-1}(y/x)$. Many computer spreadsheets and mathematical software packages implement a two-variable arctan function denoted atan2(x, y), which gives the polar angle of (x, y) in the interval $(-\pi, \pi]$. Thus,

$$\operatorname{Arg}(x + yi) = \operatorname{atan2}(x, y).$$

Given the modulus $r = |w|$ and any value of the argument $\theta = \arg(w)$ of a complex number $w = a + bi$, we have $a = r\cos\theta$ and $b = r\sin\theta$, so w can be expressed in terms of its modulus and argument as

$$w = r\cos\theta + i\, r\sin\theta.$$

The expression on the right side is called the **polar representation** of w.

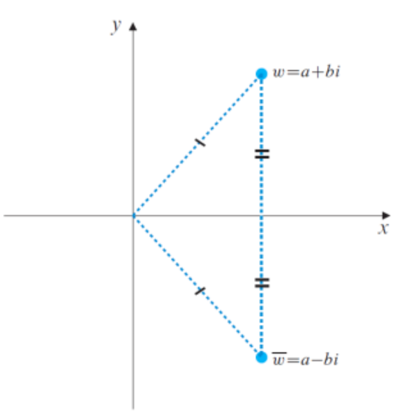


Figure I.4 A complex number and its conjugate are mirror images of each other in the real axis

DEFINITION 5

The **conjugate** or **complex conjugate** of a complex number $w = a + bi$ is another complex number, denoted $\overline{w}$, given by

$$\overline{w} = a - bi.$$

$$w\,\overline{w} = |w|^2$$

It is particularly easy to determine the product of complex numbers expressed in polar form. If

$$w = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad z = s(\cos\phi + i\sin\phi),$$

where $r = |w|$, $\theta = \arg(w)$, $s = |z|$, and $\phi = \arg(z)$, then

$$\begin{aligned} wz &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big) \\ &= rs\big(\cos(\theta+\phi) + i\sin(\theta+\phi)\big). \end{aligned}$$

(See Figure I.6.) Since arguments are only determined up to integer multiples of 2π, we have proved that

The modulus and argument of a product

$$|wz| = |w||z| \qquad \text{and} \qquad \arg(wz) = \arg(w) + \arg(z).$$

THEOREM 1

de Moivre's Theorem

$$\big(\cos\theta + i\sin\theta\big)^n = \cos n\theta + i\sin n\theta$$

den komplexa funktionen: AD/Appendix II

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The complex exponential function

$$e^z = e^x(\cos y + i \sin y) \qquad \text{for} \quad z = x + yi.$$

In particular, if $z = yi$ is pure imaginary, then

$$e^{yi} = \cos y + i \sin y,$$

a fact that can also be obtained by separating the real and imaginary parts of the Maclaurin series for e^{yi}:

$$\begin{aligned} e^{yi} &= 1 + (yi) + \frac{(yi)^2}{2!} + \frac{(yi)^3}{3!} + \frac{(yi)^4}{4!} + \frac{(yi)^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

den komplexa funktionen: egenskaper

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EXERCISES: APPENDIX II

17. Use the fact that $e^{yi} = \cos y + i \sin y$ (for real y) to show that

$$\cos y = \frac{e^{yi} + e^{-yi}}{2} \qquad \text{and} \qquad \sin y = \frac{e^{yi} - e^{-yi}}{2i}.$$

Exercise 16 suggests that we define complex functions

$$\cos z = \frac{e^{zi} + e^{-zi}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{zi} - e^{-zi}}{2i},$$

as well as extend the definitions of the hyperbolic functions to

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \text{and} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

20. Verify the identities $\cos z = \cosh(iz)$ and $\sin z = -i \sinh(iz)$. What are the corresponding identities for $\cosh z$ and $\sinh(z)$ in terms of cos and sin?

CHAPTER 9 Sequences, Series, and Power Series

Remark Observe the similarity between the series for $\sin x$ and $\sinh x$ and between those for $\cos x$ and $\cosh x$. If we were to allow complex numbers (numbers of the form $z = x + iy$, where $i^2 = -1$ and x and y are real; see Appendix I) as arguments for our functions, and if we were to demonstrate that our operations on series could be extended to series of complex numbers, we would see that $\cos x = \cosh(ix)$ and $\sin x = -i \sinh(ix)$. In fact, $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, so

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \text{and} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Such formulas are encountered in the study of functions of a complex variable (see Appendix II); from the complex point of view the trigonometric and exponential functions are just different manifestations of the same basic function, a complex exponential $e^z = e^{x+iy}$. We content ourselves here with having mentioned the interesting relationships above and invite the reader to verify them formally by calculating with series. (Such formal calculations do not, of course, constitute a proof, since we have not established the various rules covering series of complex numbers.)

komplexa tal användning för att lösa differentiella ekvationer

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18.5 Linear Differential Equations with Constant Coefficients (sammanfattning)

$$a\, y'' + b\, y' + c y = 0, \qquad (*)$$

$$ar^2 + br + c = 0. \qquad (**)$$

The auxiliary equation is quadratic and can have either

(a) two distinct real roots, r_1 and r_2 (if $b^2 > 4ac$), in which case $(*)$ has general solution $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$,

(b) a single repeated real root r (if $b^2 = 4ac$), in which case $(*)$ has general solution $y = (C_1 + C_2 t)e^{rt}$, or

(c) a pair of complex conjugate roots, $r = k \pm i\omega$ with k and ω real (if $b^2 < 4ac$), in which case $(*)$ has general solution $y = e^{kt}(C_1 \cos(\omega t) + C_2 \sin(\omega t))$.

3.7 Second-Order Linear DEs with Constant Coefficients (detaljer)

Recipe for Solving $ay'' + by' + cy = 0$

In Section 3.4 we observed that the first-order, constant-coefficient equation $y' = ky$ has solution $y = Ce^{kt}$. Let us try to find a solution of equation $(*)$ having the form $y = e^{rt}$. Substituting this expression into equation $(*)$, we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Since e^{rt} is never zero, $y = e^{rt}$ will be a solution of the differential equation $(*)$ if and only if r satisfies the quadratic **auxiliary equation**

$$ar^2 + br + c = 0, \qquad (**)$$

which has roots given by the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{D}}{2a},$$

where $D = b^2 - 4ac$ is called the **discriminant** of the auxiliary equation $(**)$.

CASE I Suppose $D = b^2 - 4ac > 0$. Then the auxiliary equation has two different real roots, r_1 and r_2, given by

$$r_1 = \frac{-b - \sqrt{D}}{2a}, \qquad r_2 = \frac{-b + \sqrt{D}}{2a}.$$

(Sometimes these roots can be found easily by factoring the left side of the auxiliary equation.) In this case both $y = y_1(t) = e^{r_1 t}$ and $y = y_2(t) = e^{r_2 t}$ are solutions of the differential equation $(*)$, and neither is a multiple of the other. As noted above, the function

$$y = A\,e^{r_1 t} + B\,e^{r_2 t}$$

CASE II Suppose $D = b^2 - 4ac = 0$. Then the auxiliary equation has two equal roots, $r_1 = r_2 = -b/(2a) = r$, say. Certainly, $y = e^{rt}$ is a solution of $(*)$. We can find the general solution by letting $y = e^{rt}u(t)$ and calculating:

$$\begin{aligned} y' &= e^{rt}\big(u'(t) + ru(t)\big), \\ y'' &= e^{rt}\big(u''(t) + 2ru'(t) + r^2u(t)\big). \end{aligned}$$

Substituting these expressions into $(*)$, we obtain

$$e^{rt}\big(au''(t) + (2ar + b)u'(t) + (ar^2 + br + c)u(t)\big) = 0.$$

Since $e^{rt} \neq 0$, $2ar + b = 0$ and r satisfies $(**)$, this equation reduces to $u''(t) = 0$, which has general solution $u(t) = A + Bt$ for arbitrary constants A and B. Thus, the general solution of $(*)$ in this case is

$$y = A\,e^{rt} + Bt\,e^{rt}.$$

CASE III Suppose $D = b^2 - 4ac < 0$. Then the auxiliary equation $(**)$ has complex conjugate roots given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = k \pm i\omega,$$

where $k = -b/(2a)$, $\omega = \sqrt{4ac - b^2}/(2a)$, and i is the imaginary unit ($i^2 = -1$;

CASE III Suppose $D = b^2 - 4ac < 0$. Then the auxiliary equation (**) has complex conjugate roots given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = k \pm i\omega,$$

where $k = -b/(2a)$, $\omega = \sqrt{4ac - b^2}/(2a)$, and i is the imaginary unit ($i^2 = -1$; see Appendix I). As in Case I, the functions $y_1^*(t) = e^{(k+i\omega)t}$ and $y_2^*(t) = e^{(k-i\omega)t}$ are two independent solutions of (*), but they are not real-valued. However, since

$$e^{ix} = \cos x + i \sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i \sin x$$

(as noted in the previous section and in Appendix II), we can find two real-valued functions that are solutions of (*) by suitably combining y_1^* and y_2^*:

$$y_1(t) = \frac{1}{2} y_1^*(t) + \frac{1}{2} y_2^*(t) = e^{kt} \cos(\omega t),$$
$$y_2(t) = \frac{1}{2i} y_1^*(t) - \frac{1}{2i} y_2^*(t) = e^{kt} \sin(\omega t).$$

Therefore, the general solution of (*) in this case is

$$y = A\, e^{kt} \cos(\omega t) + B\, e^{kt} \sin(\omega t).$$

EXAMPLE 3 Find the general solution of $y'' + 4y' + 13y = 0$.

Solution The auxiliary equation is $r^2 + 4r + 13 = 0$, which has solutions

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i.$$

Thus, $k = -2$ and $\omega = 3$. According to Case III, the general solution of the given differential equation is

$$y = A\, e^{-2t} \cos(3t) + B\, e^{-2t} \sin(3t).$$

komplexa tal användning: integraler

den 13 oktober 2020 18:03

Bestäm $\int e^x \cos x \, dx$ med hjälp av komplexa tal:

$$\int e^x \cos x \, dx = \int e^x \operatorname{Re}\{\cos x + i \sin x\} \, dx = \int e^x \operatorname{Re}\{e^{ix}\} \, dx$$

$$= \operatorname{Re}\left\{ \int e^x e^{ix} \, dx \right\} = \operatorname{Re}\left\{ \int e^{(i+1)x} \, dx \right\}$$

$$= \operatorname{Re}\left\{ \frac{e^{(i+1)x}}{i+1} + c \right\} = \operatorname{Re}\left\{ \frac{1-i}{2} e^{(i+1)x} + c \right\}$$

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2}$$

$$= \operatorname{Re}\left\{ \frac{1-i}{2} e^x (\cos x + i \sin x) + c \right\}$$

$$= \operatorname{Re}\left\{ \frac{1}{2} e^x (\cos x + \sin x) + i(\ldots) + c \right\}$$

$$= \frac{1}{2} e^x (\cos x + \sin x) + c$$

Verifiera med partiellintegration. (självstudie) !