tentamen genomgång

och lite smått och gott

några tankar

- det viktigaste lärande mål är att kunna använda litteratur (i analys) senare, några (eller många) år när kursen är ett vagt minne
- att använda litteratur alltid krävs förkunskaper (Feynman = ”infinite intelligence”)
- själv läsning är viktig (ansvar)
- några exempel följer hur boken relaterar till problem ni försökte lösa…

kontinuitet av en rationel funktion

G1 (3p). Definiera konstanten *c* och lämpliga värdet av funktionen i relevanta punkter, så att funktionen är kontinuerlig för alla reella tal *x*. Först specificera villkoret för kontinuitet och sedan visa hur du använder villkoret för att hitta c. $f(x) = \frac{-2c+(2-3c)x+3x^2}{x-3}$

Continuity at an interior point

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

There Are Lots of Continuous Functions

CHAPTER 1 Limits and Continuity

The following functions are continuous wherever they are defined:

(a) all polynomials;

(b) all rational functions;

THEOREM 6

Combining continuous functions

If the functions f and g are both defined on an interval containing c and both are continuous at c, then the following functions are also continuous at c:

1. the sum $f + g$ and the difference $f - g$;
2. the product fg;
3. the constant multiple kf, where k is any number;
4. the quotient f/g (provided $g(c) \neq 0$); and
5. the nth root $(f(x))^{1/n}$, provided $f(c) > 0$ if n is even.

Continuous Extensions and Removable Discontinuities

As we have seen in Section 1.2, a rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

$F(x)$ is continuous at $x = c$. It is called the **continuous extension** of $f(x)$ to $x = c$. For rational functions f, continuous extensions are usually found by cancelling common factors.

EXAMPLE 7 Show that $f(x) = \frac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$, and find that extension.

betydelsen av differentialer

G2 (3p). Låt u och x vara två relaterade variabler som definieras med relationen $u = \sqrt{x} + x - x^3$. Betrakta ett speciellt fall: *x=1* och *u=1*. Om vi ökar *x=1* en smula, med *dx=0.1*, alltså till *x=1+dx=1.1*, då behöver vi ändra *u* också till *1+du*. Hur mycket skall vi ändra *u* i linjär approximation, skall den öka eller minska? Alltså vad är *du* lika med? Tips: Algebra med differentialer relaterar du och dx.

DEFINITION 8

The **linearization** of the function f about a is the function L defined by

$$L(x) = f(a) + f'(a)(x-a).$$

We say that $f(x) \approx L(x) = f(a) + f'(a)(x-a)$ provides **linear approximations** for values of f near a.

EXAMPLE 2 A ball of ice melts so that its radius decreases from 5 cm to 4.92 cm. By approximately how much does the volume of the ball decrease?

Solution The volume V of a ball of radius r is $V = \frac{4}{3}\pi r^3$, so that $dV/dr = 4\pi r^2$ and $L(r + \Delta r) = V(r) + 4\pi r^2 \Delta r$. Thus,

$$\Delta V \approx L(\cancel{r+}\Delta r) = 4\pi r^2 \Delta r. \qquad \boxed{L(r+\Delta r) - L(r)}$$

Approximating Values of Functions

We have already made use of linearization in Section 2.7, where it was disguised as the formula

$$\Delta y \approx \frac{dy}{dx}\Delta x$$

and used to approximate a small change $\boxed{\Delta y = f(a+\Delta x) - f(a)}$ in the values of function f corresponding to the small change in the argument of the function from a to $a + \Delta x$. This is just the linear approximation

$$f(a + \Delta x) \approx L(a + \Delta x) = f(a) + f'(a)\Delta x.$$

EXAMPLE 1 Without using a scientific calculator, determine by approximately how much the value of $\sin x$ increases as x increases from $\pi/3$ to $(\pi/3) + 0.006$. To 3 decimal places, what is the value of $\sin((\pi/3) + 0.006)$?

Solution If $y = \sin x$, $x = \pi/3 \approx 1.0472$, and $dx = 0.006$, then

$$dy = \cos(x)\,dx = \cos\left(\frac{\pi}{3}\right)dx = \frac{1}{2}(0.006) = 0.003.$$

Thus, the change in the value of $\sin x$ is approximately 0.003, and

$$\sin\left(\frac{\pi}{3} + 0.006\right) \approx \sin\frac{\pi}{3} + 0.003 = \frac{\sqrt{3}}{2} + 0.003 = 0.869$$

rounded to 3 decimal places.

EXAMPLE 2 By approximately what percentage does the area of a circle increase if the radius increases by 2%?

2.7 Using Differentials and Derivatives

Approximating Small Changes

If one quantity, say y, is a function of another quantity x, that is,

$$y = f(x),$$

we sometimes want to know how a change in the value of x by an amount Δx will affect the value of y. The exact change Δy in y is given by

$$\Delta y = f(x + \Delta x) - f(x),$$

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\,\Delta x.$$

$$\Delta y \approx dy = f'(x)\,dx.$$

implicit derivata + kedje regel

G4 (3p). Räkna derivatan av funktionen $f(x) = \arcsin\frac{1-x}{1+x}$ med implicit derivata. Svaret går att förenkla avsevärt under visa villkor på *x*. Hitta dessa villkor och förenkla.

2.9 Implicit Differentiation

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x; in this case we know them explicitly. They are $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (see Figure 2.34), having

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x) \qquad \left(\text{The Chain Rule gives } \frac{d}{dx}y^2 = 2y\frac{dy}{dx}.\right)$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

3.5 The Inverse Trigonometric Functions

Now let us use implicit differentiation to find the derivative of the inverse sine function. If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Differentiating with respect to x, we obtain

$$1 = (\cos y)\frac{dy}{dx}.$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we know that $\cos y \ge 0$. Therefore,

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

and $dy/dx = 1/\cos y = 1/\sqrt{1 - x^2}$;

$$\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

tangent to a graph in implicit form

VG (4p). Hitta derivatan $y'(3)$ för funktionen som är definierat med den implicita formen: $x^2 + y(x)\tan(y(x) + 2) = y(x)^2 + 9$. Notera att ekvationen $x^2 + y\tan(y + 2) = y^2 + 9$ har $x = 3$ och $y = 0$ som en lösning. Räkna tangenten i denna punkt.

2.9 Implicit Differentiation

To find dy/dx by implicit differentiation:

1. Differentiate both sides of the equation with respect to x, regarding y as a function of x and using the Chain Rule to differentiate functions of y.
2. Collect terms with dy/dx on one side of the equation and solve for dy/dx by dividing by its coefficient.

EXAMPLE 3 Find $\dfrac{dy}{dx}$ if $y \sin x = x^3 + \cos y$.

Solution This time we cannot solve the equation for y as an explicit function of x, so we *must* use implicit differentiation:

$$\frac{d}{dx}(y \sin x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(\cos y) \qquad \left(\begin{array}{l}\text{Use the Product Rule}\\ \text{on the left side.}\end{array}\right)$$

$$(\sin x)\frac{dy}{dx} + y\cos x = 3x^2 - (\sin y)\frac{dy}{dx}$$

$$(\sin x + \sin y)\frac{dy}{dx} = 3x^2 - y\cos x$$

$$\frac{dy}{dx} = \frac{3x^2 - y\cos x}{\sin x + \sin y}.$$

existens av derivata

VG3 (4p). Går det att definiera konstanterna *a*, *b*, och *c* så att funktionen *f(x)* är deriverbar i *x=2*?

$$f(x) = \begin{cases} a\sin x + b & x \leq 2 \\ 2 + x & x > 2 \end{cases}$$

Om du svarar ”ja”, ange konstanterna, om du svarar ”inte” förklara varför. Visa tydligt hur du tänker. Lös det på två sätt, med direkt teknik (definition av derivatan) och den andra tekniken vi har diskuterat, och jämför.

Tayler utveckling i oändligheten

VG1 (4p). Utveckla *f(x)* i oändligheten som en potens serie. Behåll två mest dominanta termer. $f(x) = \frac{1-x}{1+3x}$. Vad betyder ”att utveckla i oändligheten”? I vilken variabel skall utvecklingen vara? Förklara tekniken.

1.3 Limits at Infinity and Infinite Limits

EXAMPLE 1 In Figure 1.16, we can see that $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$. The x-axis is a horizontal asymptote of the graph $y = 1/x$.

fact in the following examples. Example 2 shows how to obtain the limits at $\pm\infty$ for the function $x/\sqrt{x^2+1}$ by algebraic means, without resorting to making a table of values or drawing a graph, as we did above.

EXAMPLE 3 **(Numerator and denominator of the same degree)** Evaluate $\lim_{x\to\pm\infty} \dfrac{2x^2-x+3}{3x^2+5}$.

Solution Divide the numerator and the denominator by x^2, the highest power of x appearing in the denominator:

$$\lim_{x\to\pm\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\pm\infty} \frac{2-(1/x)+(3/x^2)}{3+(5/x^2)} = \frac{2-0+0}{3+0} = \frac{2}{3}.$$

EXAMPLE 2 Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Solution Rewrite the expression for $f(x)$ as follows:

$$f(x) = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} \qquad \text{Remember } \sqrt{x^2} = |x|.$$

$$= \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$$

$$= \frac{\operatorname{sgn} x}{\sqrt{1+\frac{1}{x^2}}}, \qquad \text{where } \operatorname{sgn} x = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

The factor $\sqrt{1+(1/x^2)}$ approaches 1 as x approaches ∞ or $-\infty$, so $f(x)$ must have the same limits as $x \to \pm\infty$ as does $\operatorname{sgn}(x)$. Therefore (see Figure 1.15),

$$\lim_{x\to\infty} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -1.$$

viktiga budskapen

- tentor är olika men alla har samma teman
- G
  - kontinuitet
  - ”derivarberhet”
  - gränsvärdet: L’Hopital
  - enkla integraler
  - komplexa tal
  - stenmark: att derivera med kedjeregel
  - enkla implicit derivatan (arctan, arccot, arccos)
  - differentialer (du,dx)
  - Taylor utveckling
  - enkla differentiella ekvationer
- VG
  - svårare integraler
  - svårare gränsvärdet
  - tangenten in en punkt av en graf
  - medel komplicerade differentiella ekvationer: separations teknik
  - enkla ytor med roterande grafer
- MVG
  - komplicerade gränsvärdet
  - Komplicerade derivata
  - Problem som kräver en kombination av metoder att lösa