SI-pass 5

1.1)

a) $f(x) = \dfrac{\overset{h(x)}{\cos x}}{\underset{g(x)}{2 + \sin x}}$

Kvotregeln:

The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$h'(x) = -\sin x$

$g'(x) = \cos x$

$$\Rightarrow \quad f'(x) = \frac{\overbrace{(2+\sin x)}^{g(x)} \cdot \overbrace{(-\sin x)}^{h'(x)} - \overbrace{\cos x}^{h(x)} \cdot \overbrace{\cos x}^{g'(x)}}{(2+\sin(x))^2} =$$

$$= \frac{-2\sin x - \overbrace{\sin^2 x - \cos^2 x}^{\text{Förenkla enl. trig. ettan}}}{(2+\sin(x))^2} = \left[\sin^2\theta + \cos^2\theta = 1\right] =$$

$$= \boxed{-\frac{1+2\sin x}{(2+\sin(x))^2}}$$

Kedjeregeln: $\dfrac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

b) $f(x) = \underset{f(x)}{\sin}\left(x + \underset{g(x)}{\sin}(2x)\right)$

$$\frac{d}{dx} f(x) = \boxed{\cos(x + \sin(2x)) \cdot (1 + 2\cos(2x))}$$

Glöm inte att även $g(x)$ har en inre derivata i $\sin \underbrace{2x}_{h(x)}$

c) $f(x) = x \cdot \sin^{-1}\left(\frac{1}{x}\right)$

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

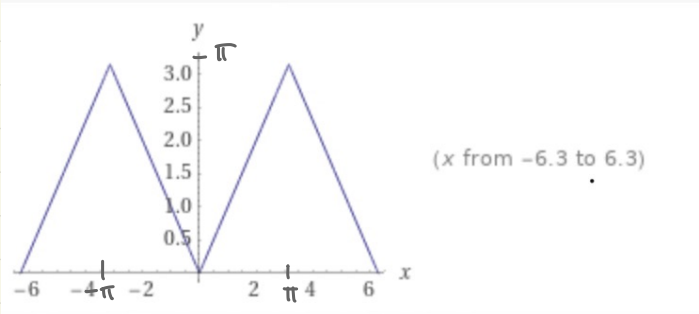
Produktregeln: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

$$f'(x) = \cancel{x} \cdot \frac{1}{\sqrt{1-\frac{1}{x^2}}} \cdot -\frac{1}{x^{\cancel{2}}} + \sin^{-1}\left(\frac{1}{x}\right) =$$

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$= \boxed{\sin^{-1}\left(\frac{1}{x}\right) - \frac{1}{x\sqrt{1-\frac{1}{x^2}}}}$$

1.2)



$f(x) = \cos^{-1}(\cos(x))$

$$\frac{d}{dx}\left(\cos^{-1}(\cos x)\right) = -\frac{1}{\sqrt{1-\cos^2 x}} \cdot (-\sin x) = \begin{cases} 1, & \sin x > 1 \\ -1, & \sin x < 1 \end{cases}$$

Deriverbar everallt utom utom i $x = n \cdot \pi$, $n = 0, 1, 2 \ldots$

2) a)

Derivatives of Inverse Functions

Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b), or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b). In either case f is one-to-one on (a, b) and has an inverse, f^{-1} there. Differentiating the cancellation identity

$$f(f^{-1}(x)) = x$$

with respect to x, using the Chain Rule, we obtain

$$f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}x = 1.$$

Thus,

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

In Leibniz notation, if $y = f^{-1}(x)$, we have $\left.\frac{dy}{dx}\right|_x = \frac{1}{\left.\frac{dx}{dy}\right|_{y=f^{-1}(x)}}$.

The slope of the graph of f^{-1} at (x, y) is the reciprocal of the slope of the graph of f at (y, x). (See Figure 3.6.)

b) $f(x) = \ln\left(\sqrt{1+x^3}\right)$ $\qquad$ $(f^{-1})'(\ln(3))$

$f^{-1}(x)$: $\quad x = \ln\left(\sqrt{1+y^3}\right) =$ $\quad$ (ty ett-till-ett) om $f'(x) > 0$

$= \frac{1}{2}\ln\left(1+y^3\right)$

$\Rightarrow \quad 2x = \ln(1+y^3)$

$\Rightarrow \quad e^{2x} = 1 + y^3$

$\Rightarrow \quad y^3 = e^{2x} - 1$

$= \left(e^{2x} - 1\right)^{1/3} \Rightarrow \boxed{f^{-1}(x) = \left(e^{2x} - 1\right)^{1/3}}$

Properties of inverse functions

1. $y = f^{-1}(x) \quad \Leftarrow \quad x = f(y)$.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.
4. $f^{-1}(f(x)) = x$ for all x in the domain of f.
5. $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f.
7. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.

$f^{-1\prime}(x) = \frac{1}{3} \cdot \frac{2e^{2x}}{(e^{2x}-1)^{2/3}}$

$\Rightarrow$

$(f^{-1})'(\ln(3)) = \frac{1}{3} \cdot \frac{2e^{2\ln 3}}{(e^{2\ln 3}-1)^{2/3}} = \frac{2 \cdot 3^{\cancel{2}}}{\cancel{3}(3^2-1)^{2/3}} = \frac{2 \cdot 3}{\underbrace{8^{2/3}}_{=4}} = \frac{3}{2}$

c)

A function f is **self-inverse** if $f^{-1} = f$, that is, if $f(f(x)) = x$ for every x in the domain of f.

$$f(x) = \frac{-2x-a}{bx-c}$$

Låt $y = f^{-1}(x) \Rightarrow \quad x = \frac{-2y-a}{by-c}$

$$\Rightarrow bxy - cx = -2y - a$$

$$\Rightarrow y(bx+2) = cx - a$$

$$\Rightarrow \quad y = \frac{cx-a}{bx+2}$$

$f^{-1}(x) = f(x)$ om $\boxed{\frac{cx-a}{bx+2} = \frac{-2x-a}{bx-c}}$

$\Rightarrow$

$\boxed{c = -2, \ a, b \text{ kan anta godtyckliga värden}}$

3.) a) $x e^y + y - 2x = \ln 2$

Implicit derivering m.a.p x:

$$\Rightarrow \quad \underbrace{x \cdot \frac{d}{dx} e^y + e^y}_{} + \frac{d}{dx} y - 2 = 0$$

Produktregeln

$$\Rightarrow \quad x e^y y' + e^y + y' - 2 = 0$$

Sätt in punkten $(1, \ln(2))$: $2y' + \cancel{2} + y' - \cancel{2} = 0$

$\Rightarrow y' = 0 \Rightarrow$ horisontell linje!

Linjens ekv: $\boxed{y = \ln(2)}$

b) $2\left(\ln\left(\frac{e}{\sqrt{x}}\right) + \ln(x)\right) = 2\left(\ln(e) - \frac{1}{2}\ln(x) + \ln(x)\right) =$

$= 2 \cdot 1 - \ln(x) + 2\ln(x) = \boxed{\ln(x) + 2}$

Veckans Quack!

$$f(x) = a^x$$

$$f'(0) = k \quad \Rightarrow \quad f'(x) = k \cdot f(x)$$

Vi använder derivatans def. :

The **derivative** of a function f is another function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} =$$

$$= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{f'(0) = k} = a^x k = \boxed{k f(x)} \quad \text{V.S.V}$$