MVE136 Random Signals Analysis

Written exam Monday 15 August 2022 2–6 PM

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Aids: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Motivations: All answers/solutions must be motivated. Good luck!

Task 1. Let $\{X(t)\}_{t\geq 0}$ be a zero-mean Gaussian process with autocorrelation function $R_{XX}(s,t)=\min(s,t)$. Find the variance of the random variable $Y=X(1)+\int_0^1 X(s)\,ds$. **(5 points)**

Task 2. Show that if for a discrete time Markov chain state i is recurrent and does not communicate with state j, that is, $i \not\leftrightarrow j$, then $p_{ij}=0$. **(5 points)**

Task 3. Customers arrive at a bank as a Poisson process with rate $\lambda>0$ per hour. Suppose two customers arrived during the first hour. What is the probability that

(a) both arrived during the first 20 minutes (=1/3 hour)? **(2,5 points)**

(b) at least one arrived during the first 20 minutes (=1/3 hour)? **(2,5 points)**

Task 4. Let $\{X(k)\}_{k\in\mathbb{Z}}$ be a discrete time WSS random process. Show that $Y(k)=X(k)^4$, $k\in\mathbb{Z}$, need not be a WSS process. (Hint: Find a WSS $X(k)$-process such that the distribution of the $X(k)$ random process values vary with k and therefore also the values of $E[X(k)^4]$. You might want to use that $E[Y^4]=3$ for Y a standard normal random variable.) **(5 points)**

Task 5. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function

$$R_{XX}(\tau)=\frac{A\omega_0}{\pi}\,\frac{\sin(\omega_0\tau)}{\omega_0\tau}\quad\text{for }\tau\in\mathbb{R},$$

where $A,\omega_0>0$ are constants, whilst the LTI system has impulse response of the same form

$$h(t)=\frac{\omega_1}{\pi}\,\frac{\sin(\omega_1 t)}{\omega_1 t}\quad\text{for }t\in\mathbb{R},$$

where $\omega_1>0$ is a constant. Find the autocorrelation function of the output $Y(t)$ from the system. **(5 points)**

Task 6. Derive the Wiener-Hopf equations. **(5 points)**

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Solutions to written exam August 2022

Task 1. As both $X(1)$ and $\int_0^1 X(s)\,ds$ are zero-mean we have $\mathbf{Var}\{X(1)+\int_0^1 X(s)\,ds\}$ $= \mathbf{E}\{X(1)^2\}+2\,\mathbf{E}\{X(1)\int_0^1 X(s)\,ds\}+\mathbf{E}\{(\int_0^1 X(r)\,dr)\,((\int_0^1 X(s)\,ds)\} = R_{XX}(1,1)+2\int_0^1$ $R_{XX}(1,s)\,ds+\int_0^1\int_0^1 R_{XX}(r,s)\,drds = 1+2\int_0^1 s\,ds+\int_0^1\int_0^1 \min(r,s)\,drds = 1+1+1/3.$

Task 2. If $p_{ij}>0$, then $p_{ji}(n)=0$ for all n as otherwise i and j would communicate. But then the process starting in i has a probability at least $p_{ij}>0$ of never returning to i which contradicts the recurrence of i.

Task 3. (a) $\mathbf{P}\{X(1/3)=2|X(1)=2\} = \frac{\mathbf{P}\{X(1/3)=2,\,X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}}$
$= \frac{\mathbf{P}\{X(1/3)=2\}\,\mathbf{P}\{X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}} = \frac{((\lambda/3)^2/2!)\,\mathrm{e}^{-\lambda/3}\,((2\lambda/3)^0/0!)\,\mathrm{e}^{-2\lambda/3}}{(\lambda^2/2!)\,\mathrm{e}^{-\lambda}} = \frac{1}{9}.$
(b) $\mathbf{P}\{X(1/3)\geq 1|X(1)=2\} = \frac{\mathbf{P}\{X(1/3)=1,\,X(1)-X(1/3)=1\}}{\mathbf{P}\{X(1)=2\}} + \frac{\mathbf{P}\{X(1/3)=2,\,X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}}$
$= \frac{\mathbf{P}\{X(1/3)=1\}\,\mathbf{P}\{X(1)-X(1/3)=1\}}{\mathbf{P}\{X(1)=2\}} + \frac{\mathbf{P}\{X(1/3)=2\}\,\mathbf{P}\{X(1)-X(1/3)=0\}}{\mathbf{P}\{X(1)=2\}} = \ldots = \frac{4}{9}+\frac{1}{9}=\frac{5}{9}.$

Task 4. Let $X(k)=N_k$ for k even and $X(k)=e_k$ for k odd where $\{N_k\}$ and $\{e_k\}$ are independent random variables with N_k $\mathrm{N}(0,1)$-distributed and e_k a random sign (1 or -1 with probability 1/2 each). Then $\{X(k)\}$ is unit variance discrete time white noise and thus WSS. However, $E[X(k)^4]=3$ for k even while $E[X(k)^4]=1$ for k odd so that $\{X(k)^4\}_{k\in\mathbb{Z}}$ is not WSS.

Task 5. We have $S_{XX}(f) = ((A\omega_0)/\pi)\,\mathrm{rect}(f/\omega_0)/\omega_0 = A\,\mathrm{rect}(f/\omega_0)/\pi$ and similarly $H(f) = \mathrm{rect}(f/\omega_1)/\pi$, so that $S_{YY}(f) = |H(f)|^2 S_{XX}(f) = A\,\mathrm{rect}(f/\omega_1)^2\,\mathrm{rect}(f/\omega_0)$ $/\pi^3 = A\,\mathrm{rect}(f/\min\{\omega_0,\omega_1\})/\pi^3$ which corresponds to

$$R_{YY}(\tau) = \frac{A\,\min\{\omega_0,\omega_1\}}{\pi^3}\,\frac{\sin(\min\{\omega_0,\omega_1\}\,\tau)}{\min\{\omega_0,\omega_1\}\,\tau} \quad \text{for } \tau\in\mathbb{R}.$$

Task 6. See the document L14.pdf authored by Tomas McKelvey on the course Canvas web page from Fall 2020.