MVE136 Random Signals Analysis

Written exam Tuesday 3 January 2023 2–6 PM

TEACHER AND TELEPHONE JOUR: Patrik Albin 031 7723512.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate $\Pr(X(1)Y(2) > 0)$ when $X(t)$ and $Y(t)$, $t \in \mathbb{R}$, are independent zero-mean WSS Gaussian processes with PSD's $S_{XX}(f) = S_{YY}(f) = \mathrm{e}^{-|f|}$. **(5 points)**

Task 2. Calculate $\Pr(X(1)Y(2)Z(3) = 4)$ when $X(t)$, $Y(t)$ and $Z(t)$ are independent Poisson processes with intensity/rate 1. **(5 points)**

Task 3. Is the process $\sin(X(t)^2)$ WSS when $\{X(t)\}_{t\in\mathbb{R}}$ is a WSS Gaussian process? **(5 points)**

Task 4. A Markov chain has four states $\{0,1,2,3\}$ and all transition probabilities $p_{ij} = 1/4$. Calculate the expected value of the time it takes for the chain to move from state 0 to state 3. **(5 points)**

Task 5. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function

$$R_{XX}(\tau) = \frac{A\omega_0}{\pi}\,\frac{\sin(\omega_0\tau)}{\omega_0\tau} \quad \text{for } \tau \in \mathbb{R},$$

where $A, \omega_0 > 0$ are constants, whilst the LTI system has impulse response of the same form

$$h(t) = \frac{\omega_1}{\pi}\,\frac{\sin(\omega_1 t)}{\omega_1 t} \quad \text{for } t \in \mathbb{R},$$

where $\omega_1 > 0$ is a constant. Find the autocorrelation function of the output $Y(t)$ from the system. **(5 points)**

Task 6. As you know the Wiener filter formula for filtration of a noise disturbed signal $X(t) = Z(t) + N(t)$ in order to optimally reconstruct $Z(t)$ is $H(f) = \frac{S_{ZZ}(f)}{S_{ZZ}(f)+S_{NN}(f)}$ when the signal $Z(t)$ and the noise $N(t)$ are independent and zero-mean WSS. How does this formula change if $Z(t)$ and $N(t)$ are dependent and jointly zero-mean WSS with crosspectraldensity $S_{ZN}(f)$? **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam January 2023

Task 1. $\Pr(X(1)Y(2)>0) = \Pr(X(1)>0)\Pr(Y(2)>0) + \Pr(X(1)<0)\Pr(Y(2)<0)$ $= \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$ since $X(1)$ and $Y(2)$ are independent zero-mean normaldistributed with variance $\int_{-\infty}^{\infty} \mathrm{e}^{-|f|}\,df > 0$.

Task 2. $\Pr(X(1)Y(2)Z(3)=4) = \Pr(X(1)=4)\cdot\Pr(Y(2)=1)\cdot\Pr(Z(3)=1)+\Pr(X(1)=1)\cdot\Pr(Y(2)=4)\cdot\Pr(Z(3)=1)+\Pr(X(1)=1)\cdot\Pr(Y(2)=1)\cdot\Pr(Z(3)=4)+\Pr(X(1)=2)\cdot\Pr(Y(2)=2)\cdot\Pr(Z(3)=1)+\Pr(X(1)=2)\cdot\Pr(Y(2)=1)\cdot\Pr(Z(3)=2)+\Pr(X(1)=2)\cdot\Pr(Y(2)=2)\cdot\Pr(Z(3)=2) = \Pr(\mathrm{Po}(1)=4)\cdot\Pr(\mathrm{Po}(2)=1)\cdot\Pr(\mathrm{Po}(3)=1)+\Pr(\mathrm{Po}(1)=1)\cdot\Pr(\mathrm{Po}(2)=4)\cdot\Pr(\mathrm{Po}(3)=1)+\Pr(\mathrm{Po}(1)=1)\cdot\Pr(\mathrm{Po}(2)=1)\cdot\Pr(\mathrm{Po}(3)=4)+\Pr(\mathrm{Po}(1)=2)\cdot\Pr(\mathrm{Po}(2)=2)\cdot\Pr(\mathrm{Po}(3)=1)+\Pr(\mathrm{Po}(1)=2)\cdot\Pr(\mathrm{Po}(2)=1)\cdot\Pr(\mathrm{Po}(3)=2)+\Pr(\mathrm{Po}(1)=2)\cdot\Pr(\mathrm{Po}(2)=2)\cdot\Pr(\mathrm{Po}(3)=2)=\ldots$.

Task 3. When $X(t)$ is WSS Gaussian then $X(t)$ is also strictly stationary, i.e., the finite dimensional distributions of $X(t)$ are translation invariant. But then the finite dimensional distributions of $\sin(X(t)^2)$ are also translation invariant, i.e., $\sin(X(t)^2)$ is strictly stationary and therefore also WSS.

Task 4. For the sought after expectation E we have $E = 1 + (3/4)\cdot E$ giving $E = 4$.

Task 5. We have $S_{XX}(f) = ((A\omega_0)/\pi)\,\mathrm{rect}(f/\omega_0)/\omega_0 = A\,\mathrm{rect}(f/\omega_0)/\pi$ and similarly $H(f) = \mathrm{rect}(f/\omega_1)/\pi$, so that $S_{YY}(f) = |H(f)|^2 S_{XX}(f) = A\,\mathrm{rect}(f/\omega_1)^2\,\mathrm{rect}(f/\omega_0)$ $/\pi^3 = A\,\mathrm{rect}(f/\min\{\omega_0,\omega_1\})/\pi^3$ which corresponds to

$$R_{YY}(\tau) = \frac{A\min\{\omega_0,\omega_1\}}{\pi^3}\,\frac{\sin(\min\{\omega_0,\omega_1\}\tau)}{\min\{\omega_0,\omega_1\}\tau} \quad \text{for } \tau\in\mathbb{R}.$$

Task 6. A straightforward modification of the derivation of the Wiener filter gives $H(f)$ $= \frac{S_{ZZ}(f)+S_{ZN}(f)}{S_{ZZ}(f)+S_{NN}(f)+2S_{ZN}(f)}$.