L13: Spectrum estimation – nonparametric and parametric

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Learning objectives

After today's lecture you should be able to

- Describe what the **periodogram** is and list its most important properties.
- Explain why the **modified periodogram** is sometimes an important improvement.
- Describe why **Blackman-Tukey's method** has lower variance than the periodogram.
- Derive how to **estimate AR-parameters** from data.

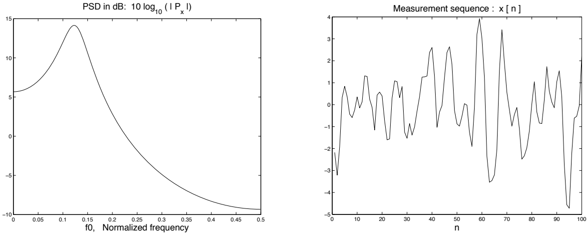
$\rightsquigarrow$ abilities that will help you to understand laboration 2!

Notation and objective

- Let $x[n]$ be a zero mean, wide sense stationary process.
- **Objective**: estimate $P_x\left(e^{j\omega}\right)$ from a measurement sequence $x[0], x[1], \ldots, x[N-1]$.
- **Application examples**: speech processing, medical diagnosis, control system design, radar, etc.

Today's main example

- **Example 1**: $x[n]$ is an AR(2)-process and thus a WSS-process



  – **Task:** estimate $P_x(e^{j\omega})$ from $x[n]$.

The periodogram: definition

- The periodogram is defined as

$$\hat{P}_{per}\left(e^{j\omega}\right) = \frac{1}{N}\left|X_N\left(e^{j\omega}\right)\right|^2$$

  where

$$X_N\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}.$$

  is the DTFT of the measured sequence.

- **Note**:
  1. Efficiently computed using DFT (FFT) (to yield discrete samples).
  2. Asymptotically unbiased!

The periodogram vs the PSD

- The PSD can be defined as

$$P_x\left(e^{j\omega}\right) = \lim_{N\to\infty} \frac{1}{N} E\left\{\left|X_N\left(e^{j\omega}\right)\right|^2\right\}$$

  where $X_N\left(e^{j\omega}\right)$ is as above.

- Two practical difficulties
  1. We are only given one sequence $\Rightarrow$ can't do expected values.
  2. Given a finite sequence (N observations) $\Rightarrow$ can't let $N \to \infty$.

- The periodogram ignores both of these!

The periodogram: bias for finite N

- The DFT $X_N\left(e^{j\omega}\right)$ can be written as

$$X_N\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} w_R[n]x[n]e^{-j\omega n}$$
$$= \mathrm{DTFT}(w_R[n]x[n]) = W_R\left(e^{j\omega}\right) \star X\left(e^{j\omega}\right).$$

- $W_R\left(e^{j\omega}\right)$ changes the PSD and creates two types of bias:
  1. The mainlobe smears out the peaks $\Rightarrow$ frequency resolution is limited by $\Delta\omega \approx 2\pi/N$.
  2. Sidelobes cause "frequency masking and leaking".

- As $N \to \infty$, $W_R\left(e^{j\omega}\right) \star X\left(e^{j\omega}\right) \to X\left(e^{j\omega}\right)$ which means that the bias disappears.

The modified periodogram

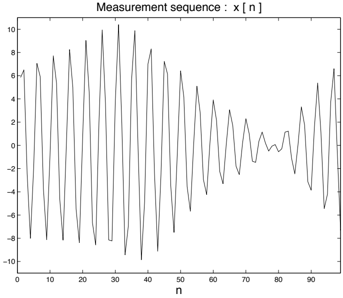
- **Idea**: replace $w_R[n]$ by a different window
- Among all windows with the same width N, the rectangular window has the narrowest mainlobe (which is a benefit).
- Examples of other windows are Bartlett (triangular), Hamming and Chebyshev.
  All of these have lower sidelobes but wider mainlobes.

The modified periodogram – example 2

- Here $x[n]$ is a deterministic function:

$$x[n] = A_1 \cos(0.2 \cdot 2\pi n) + A_1 \sin(0.21 \cdot 2\pi n) + A_2 \cos(0.2295 \cdot 2\pi n).$$

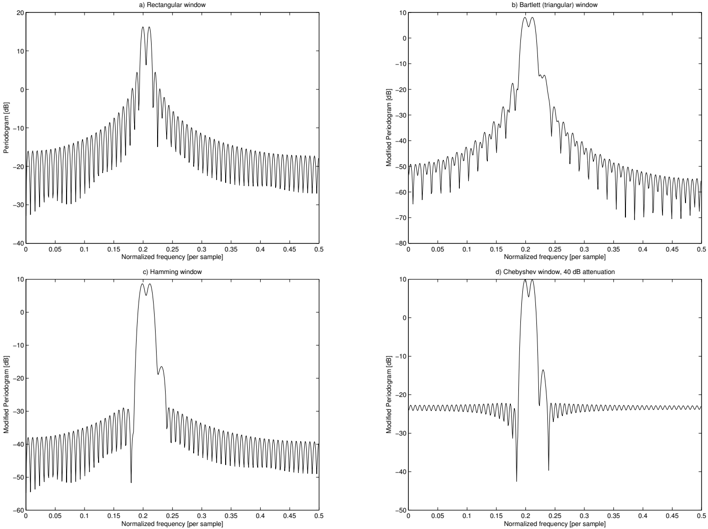
where $A_2 < A_1 \qquad (20 \log_{10}(A_1/A_2) = 26).$



- **Note:** $x[n]$ is not a WSS signal. Still useful to illustrate situations where resolution is important.

The modified periodogram – example 2

- Four common windows evaluated for example 2:

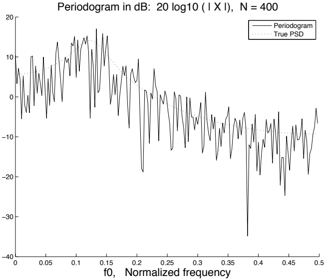


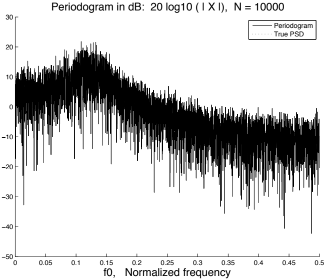
Blackman-Tukey's method

- For large N it holds that

$$\mathrm{Var}(\hat{P}_{per}(e^{j\omega})) \approx P_x^2(e^{j\omega})$$

which means that $\hat{P}_{per}(e^{j\omega})$ is **very noisy**.





- We would like to perform **averaging** in order to **reduce variance**!

Blackman-Tukey's method

- Here we only discuss Blackman-Tukey's method which is often a good choice.
  $\rightsquigarrow$ see lecture notes for Bartlett's and Welch's methods.

- **Important observation:** $X_N\left(e^{j\omega_1}\right)$ and $X_N\left(e^{j\omega_2}\right)$ are approximately uncorrelated whenever $N|\omega_1 - \omega_2|$ is "large".

  Idea: perform local averaging of the periodogram,

$$\hat{P}_{BT}\left(e^{j\omega}\right) = \frac{1}{2\pi} W_{lag}\left(e^{j\omega}\right) \star \hat{P}_{per}\left(e^{j\omega}\right)$$

  where $W_{lag}\left(e^{j\omega}\right)$ is centred around $\omega \approx 0$.

- Loosely speaking, $W_{lag}\left(e^{j\omega}\right)$ is a rectangular box.

Blackman-Tukey's method

- BT's method can also be explained from

$$\hat{P}_{per}(e^{j\omega}) = \text{DTFT}\left\{\hat{r}_x[n]\right\}.$$

- Taking the inverse transform of $\hat{P}_{BT}\left(e^{j\omega_1}\right)$ gives

$$\hat{r}_{BT}[n] = w_{lag}[n]\hat{r}_x[n]$$

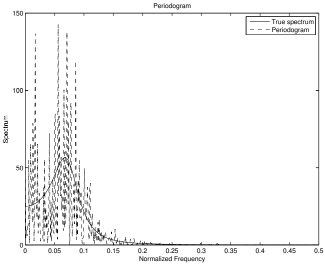
$\rightsquigarrow$ estimates $\hat{r}_x[n]$ which are less reliable are given a smaller weight $w_{lag}[n]$!

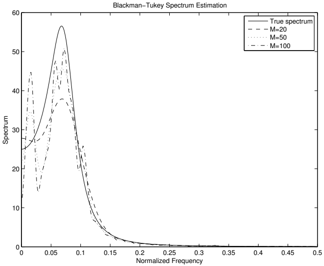
- A simple weight function is the triangular window (used below):

$$w_{lag}[n] = \begin{cases} \frac{M-|n|}{M} & \text{if } |n| \leq M \\ 0 & \text{otherwise.} \end{cases}$$

Blackman-Tukey's method

- One can show that BT **reduces variances** but **increases bias**.

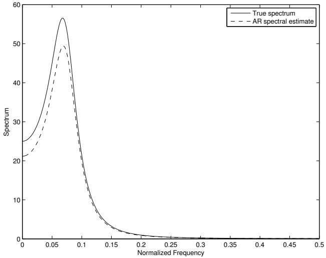




- Small M: 1) gives a wide window $W_{lag}(e^{j\omega})$ 2) makes $P_{BT}(e^{j\omega})$ smooth 3) gives a bias in $P_{BT}(e^{j\omega})$, especially where $P_x(e^{j\omega})$ varies quickly.

An illustration

- The AR-parameters can be estimated using the normal equations.
- Works very well for above example:



$\rightsquigarrow$ low variance since we only have two parameters to estimate!
$\rightsquigarrow$ bias may be large if spectrum is not $\approx$ AR.

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