L11: ACF, PSD, AR, MA and ARMA

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- 4 lectures
  - Course material: Compendium by Mats Viberg (on Canvas)
- Lab project 2, more details on Canvas

Ch 2 & 3 ACF, PSD, AR, MA, ARMA
- Signal models

Ch 4 Spectral estimation, non-parametric and parametric.
- Characterization and signal model estimation

Ch 5 Optimal filters
- Signal estimation

Why study?

- Signal models
  - Understand given signal descriptions
  - Need ways to describe signals
  - With the right framework we can develop tools
    - Signal characterization
    - Optimal filtering
- Signal model estimation
  - Ways to characterize the signals and the systems that generated it
    - Example: Monitoring the structural health of a bridge using vibration analysis.
  - Predict future signal values
    - Share prices on the stock market
    - Electric energy consumption
- Optimal filters
  - Suppress noise
  - Signal separation
  - Estimate states/parameters that are not directly observed

Fields where this is important

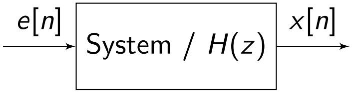
- Communication
- Monitoring
- Control
- Economy / Econometrics
- Meteorology
- Signal Processing
- Diagnostic medicin
- Biology etc.

After today's lecture you should be able to

- Describe what an **autocorrelation function** (ACF) is (definition and interpretation).
- Explain why the Fourier transform of the ACF is called the **power spectral density** (PSD)
- Summarize what **AR, MA and ARMA processes** are.

Discrete time systems and signals

Focus is on *discrete time (DT)* signals and systems.



We regard $e[n]$ and $x[n]$ as stochastic processes

We regard $H(z)$ is a linear system/filter

n is integer valued, often a time index.

Standing assumptions:

- Signals have zero mean value, $\mathbf{E}\{e[n]\} = 0$, $\mathbf{E}\{x[n]\} = 0$
- Signal are *wide sense stationary (WSS)*, which implies
  - $\mathbf{E}\{x[n]x[n-k]\} = r_x[k]$
  - First and second order signal properties are invariant w.r.t. absolute time

What is a strict stationary process?

Autocorrelation function (ACF)

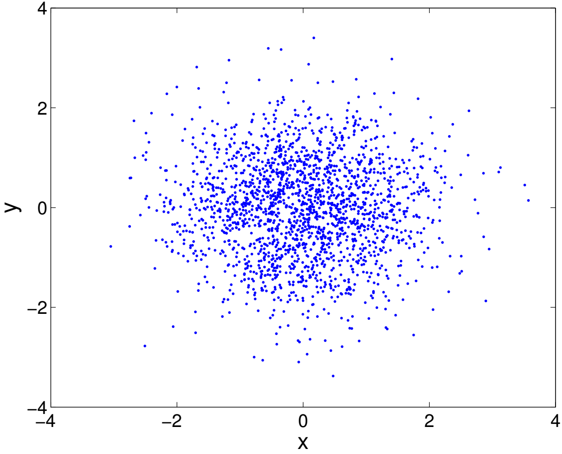
$$r_x[k] = \mathbf{E}\{x[n]x[n-k]\}$$

Example: If $r_x[0] = 1$ what do we know about $x[n]$.

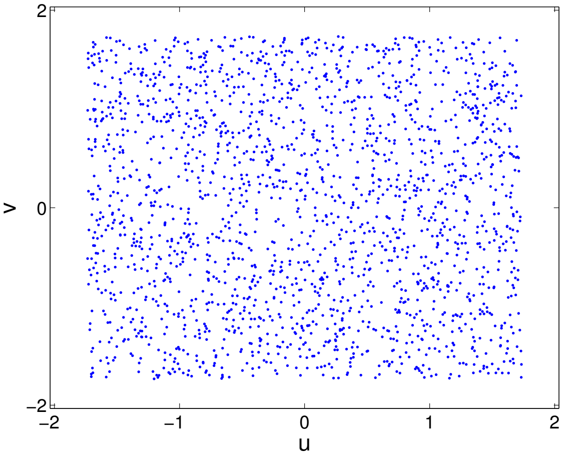
Suppose $x[n]$ is ergodic

$$r_x[0] = \mathbf{E}\{x^2[n]\} \approx \frac{1}{N}\sum_{n=1}^{N} x^2[n]$$

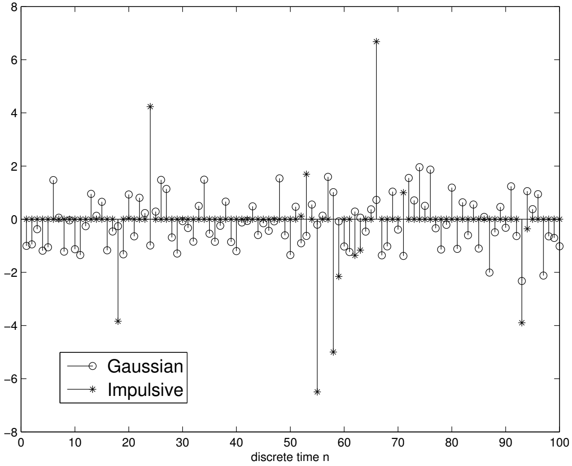
- What is $\mathrm{E}\left(x^2\right)$, $\mathrm{E}\left(y^2\right)$ and $\mathrm{E}\left(x\,y\right)$?



- What is $\mathrm{E}\left(v^2\right)$, $\mathrm{E}\left(u^2\right)$ and $\mathrm{E}\left(u\,v\right)$?



- What is $r_x[0]$ for the two processes?



ACF

$$r_x[0] = \mathbf{E}\{x^2[n]\} \quad \text{is the variance}$$

ACF gives no information about the underlying density!

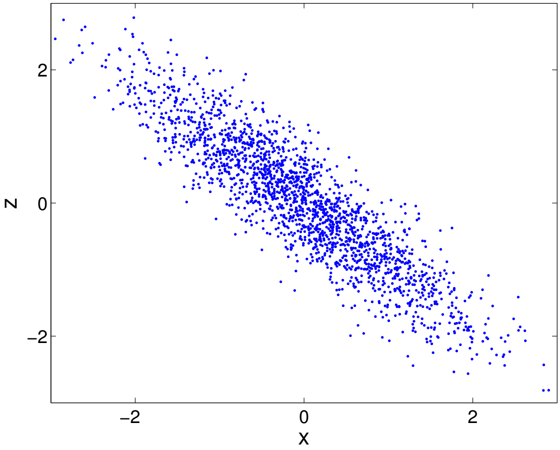
Example: Interpretation of $r_x[k] > 0$.

$\Rightarrow \quad x[n], x[n-k]$ often has the same sign

$$r_x[k] \approx \frac{1}{N}\sum_{n=1}^{N} x[n]x[n-k]$$

$r_x[k]$ give information about how the signal repeats certain patterns

- What is $\mathrm{E}\left(x^2\right)$, $\mathrm{E}\left(z^2\right)$ and $\mathrm{E}\left(x\,z\right)$?



Power Spectral Density (PSD)

The PSD of $x[n]$ is the discrete time Fourier transform (DTFT) of the ACF

$$P_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} r_x[k] e^{-j\omega k} \quad \omega \text{ is in rad/sample}$$

where the ACF is

$$r_x[k] = \mathbf{E}\{x[n]x[n-k]\}$$

and

$$r_x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega}) e^{j\omega k} \, d\omega$$

Why the name power spectral density?

$$\begin{aligned} \text{Power of } x[n] &= \mathbf{E}\{x^2[n]\} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_x(e^{j\omega}) \, d\omega \end{aligned}$$

$P_x(e^{j\omega})$ is power per rad/sample

1. If $r_x[n]$ and $P_x(z)$ are each others Z-transform pairs, then

$$P_x(e^{j\omega}) = P_x(z)|_{z=e^{j\omega}}$$

2. If $e[n]$ is the input to a linear system with impulse response $h[n]$ then (linear convolution)

$$x[n] = \sum_{k=-\infty}^{\infty} h[k]e[n-k]$$

$$X(z) = H(z)E(z) \quad \text{if these exists}$$

$$P_x(e^{j\omega}) = |H(e^{j\omega})|^2 P_e(e^{j\omega})$$

We have the Z-transform pairs

$$
\begin{aligned}
r_x[n-1] &\leftrightarrow z^{-1} P_x(z) \\
r_x[n-m] &\leftrightarrow z^{-m} P_x(z) \\
x[n] &\leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n] \\
\delta[n] &\leftrightarrow 1
\end{aligned}
$$

where the Kronecker delta function is

$$
\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}
$$

Signal models

$$\xrightarrow{e[n]} \boxed{H(z)} \xrightarrow{x[n]}$$

$e[n]$ is WSS white noise $\Rightarrow$

$$\mathbf{E}\{e[n]\} = 0$$

$$r_e[k] = \mathbf{E}\{e[n]e[n-k]\} = \sigma_e^2 \delta[k] = \begin{cases} \sigma_e^2 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$P_e(e^{j\omega}) = \text{DTFT}\{r_e[k]\} = \sigma_e^2 \, \text{DTFT}\{\delta[k]\} = \sigma_e^2$$

white = all frequencies have equal power. Output PSD is shaped by the filter

$$P_x(e^{j\omega}) = |H(e^{j\omega})|^2 \sigma_e^2$$

AR process

$x[n]$ is an AR(p) process if

$$x[n] + a_1 x[n-1] + \ldots + a_p x[n-p] = e[n]$$

or (in it's generative form)

$$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + e[n]$$

the "new value" is the regression of the old values plus the innovation. Hence the name auto-regression AR.
The Z-transform is (if exists)

$$X(z)(1 + a_1 z^{-1} + \ldots + a_p z^{-p}) = E(z)$$

which yields

$$H(z) = \frac{X(z)}{E(z)} = \frac{1}{1 + a_1 z^{-1} + \ldots a_p z^{-p}}$$

$x[n]$ is an MA(q) process if

$$x[n] = e[n] + b_1 e[n-1] + \ldots + b_q e[n-q]$$

The "new value" is a moving average (MA) of the input process $e[n]$
The Z-transform is

$$H(z) = 1 + b_1 z^{-1} + \ldots + b_q z^{-q}$$

ARMA process

$x[n]$ is an ARMA(p, q) process if

$$x[n]+a_1x[n-1]+\ldots+a_px[n-p] = e[n]+b_1e[n-1]+\ldots+b_qe[n-q]$$

or (in it's generative form)

$$x[n] = -\sum_{k=1}^{p} a_k x[n-k] + e[n] + \sum_{k=1}^{q} b_q e[n-q]$$

the "new value" is the regression of the old values plus moving average of the innovation. Hence the name auto-regression moving average ARMA.

$$H(z) = \frac{1 + b_1 z^{-1} + \ldots + b_q z^{-q}}{1 + a_1 z^{-1} + \ldots + a_p z^{-p}}$$