Lecture 12

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Agenda today

- How to determine the ACF/PSD for the signal models
- What type of ACF/PSD can these models describe

Learning objectives: L12

After today's lecture you should be able to

- Sketch the **PSD of an AR, MA and ARMA** model given poles and zeros of the transfer function.
- Calculate the **ACF for an MA** process.
- Derive the **Yule-Walker** equations used to compute the **ACF of an AR process**
- **Estimate an AR process from data**.

Signal models: AR, MA and ARMA processes

A stochastic model for $x[n]$

$$e[n] \longrightarrow \boxed{\quad h[n] \quad} \longrightarrow x[n]$$

where $e[n]$ is WSS, white noise, $\Rightarrow \mathrm{E}\{e[n]\} = 0$ and

$$r_e[k] = \mathrm{E}\{e[n]e[n+k]\} = \delta[k]\sigma_e^2 = \begin{cases} \sigma_e^2 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

- $P_e(e^{j\omega}) = DTFT\{r_e[k]\} = \sigma_e^2 DTFT\{\delta[k]\} = \sigma_e^2$.
- $H(e^{j\omega}) = DTFT\{h[k]\}$
- $P_x(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_e^2$

The filter $H(z)$ shape the spectrum of the process $x[n]$.

Definition of ARMA

- $x[n]$ is **an ARMA(p,q)-process if**

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} = \frac{b(z)}{a(z)}$$

  or, equivalently, if

$$x[n] + \cdots + a_p x[n-p] = e[n] + \cdots + b_q e[n-q] \qquad (1)$$

- Important special cases:
  1) $x[n]$ is AR(p) if $B(z) = 1$,
  2) $x[n]$ is MA(q) if $A(z) = 1$

PSD of ARMA

The PSD of an ARMA

$$e[n] \longrightarrow \boxed{\; h[n] \;} \longrightarrow x[n]$$

- We get

$$P_x\left(e^{j\omega}\right) = P_e\left(e^{j\omega}\right)\left|H\left(e^{j\omega}\right)\right|^2 = \sigma_e^2\left|H\left(e^{j\omega}\right)\right|^2$$

$\rightsquigarrow$ the filter $H(z)$ shapes the PSD of $x[n]$.

- Note: $e[n]$ is white noise

$$\Rightarrow \quad r_e[n] = \delta[n]\sigma_e^2 \longleftrightarrow P_e\left(e^{j\omega}\right) = \sigma_e^2.$$

ACF of ARMA

- **Techniques to compute the ACF:**

1) Find the PSD as above and compute

$$r_x[n] = IDTFT\left\{P_x\left(e^{j\omega}\right)\right\}$$

$\rightsquigarrow$ general solution, but often rather complicated.

2) Derive $h[n] = IDTFT\left\{H\left(e^{j\omega}\right)\right\}$ and compute

$$r_x[n] = \sigma_e^2\, h[n] \star h[-n] = \sum_{k=-\infty}^{\infty} h[k]h[k-n]$$

$\rightsquigarrow$ another possible, but complicated, alternative.

3) Derive linear equations from the difference equation (1)

   1. $\rightsquigarrow$ a simple solution!
   2. Only works for AR and MA processes.
   3. Enables us to estimate AR-parameters from data.

- Consider an AR(p)-process

$$x[n] + a_1 x[n-1] + \cdots + a_p x[n-p] = e[n]$$

- Multiply by $x[n-k]$, $k \geq 0$, and take expectations:

Yule-Walker (YW) equations

$$r_x[k] + a_1 r_x[k-1] + \cdots + a_p r_x[k-p] = \sigma_e^2 \, \delta[k]$$

- Note: 1) YW are linear in the autocorrelation function, $r_x[n]$
  2) by changing k, we can collect any number of equations.
  $\Rightarrow$ easy to find $r_x[n]$ using YW