MVE 136 L12

Ex AR(1), $\sigma_e^2 = 1$

$$x(n) + 0{,}5\, x(n-1) = e(n)$$

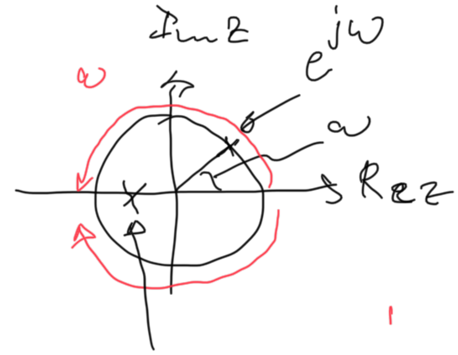
$$Z: \quad X(z) + 0{,}5 \cdot z^{-1} X(z) = E(z)$$

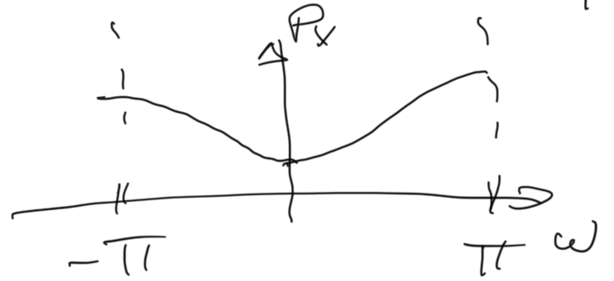
$$(1 + 0{,}5 z^{-1})\, X(z) = E(z)$$

$$H(z) = \frac{X(z)}{E(z)} = \frac{1}{1+0{,}5 z^{-1}} = \frac{z}{z+0{,}5}$$

$\Rightarrow$ pole at $z = -0{,}5$

$$P_x(e^{j\omega}) = |H(z)|^2_{z=e^{j\omega}} = \frac{1}{|e^{j\omega} + 0{,}5|^2}$$





Ex MA(1), $\sigma_e^2 = 1$

$$x(n) = e(n) - 0{,}5\, e(n-1)$$

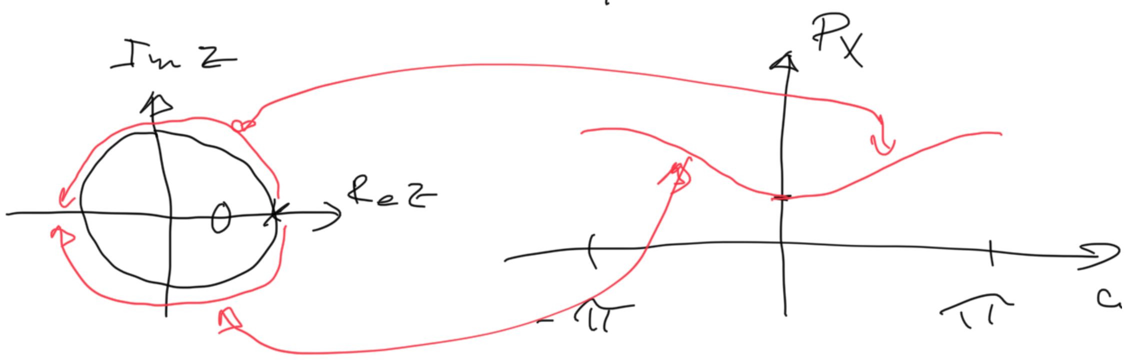
$$Z: \quad X(z) = E(z) - 0{,}5\, E(z) \cdot z^{-1}$$

$$X(z) = (1 - 0{,}5 z^{-1})\, E(z)$$

$$\Rightarrow H(z) = \frac{X(z)}{E(z)} = 1-0{,}5\, z^{-1}$$

$\Rightarrow$ zero at $z = 0{,}5$

$$P_X(e^{j\omega}) = \left|H(e^{j\omega})\right|^2 = \left|1-0{,}5\, e^{-j\omega}\right|^2$$



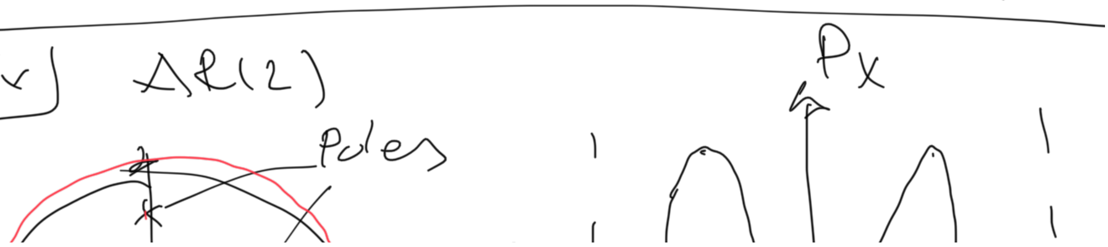
PSD for ARMA(p,q)

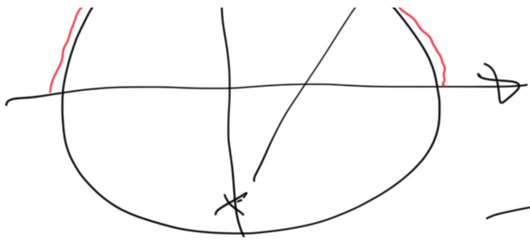
$$P_X(e^{j\omega}) = \sigma_e^2 \left|H(e^{j\omega})\right|^2 = \sigma_e^2 \left|\frac{B(e^{j\omega})}{A(e^{j\omega})}\right|^2$$

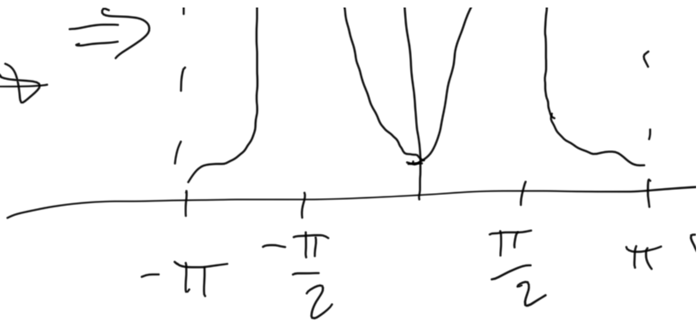
$$= \sigma_e^2 \left.\frac{\prod_{i=1}^{q} \left|1 - z_i z^{-1}\right|^2}{\prod_{i=1}^{p} \left|1 - p_i z^{-1}\right|^2}\right|_{z=e^{j\omega}}$$

$$= \sigma_e^2 \frac{\prod \text{distance}^2 \text{ between } e^{j\omega} \text{ \& zer}}{\prod \text{dist}^2 \text{ between } e^{j\omega} \text{ and pole}}$$

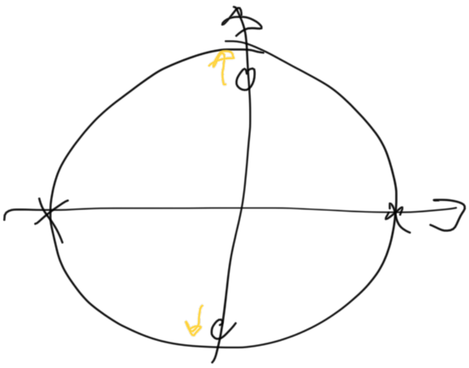
Ex) AR(2)

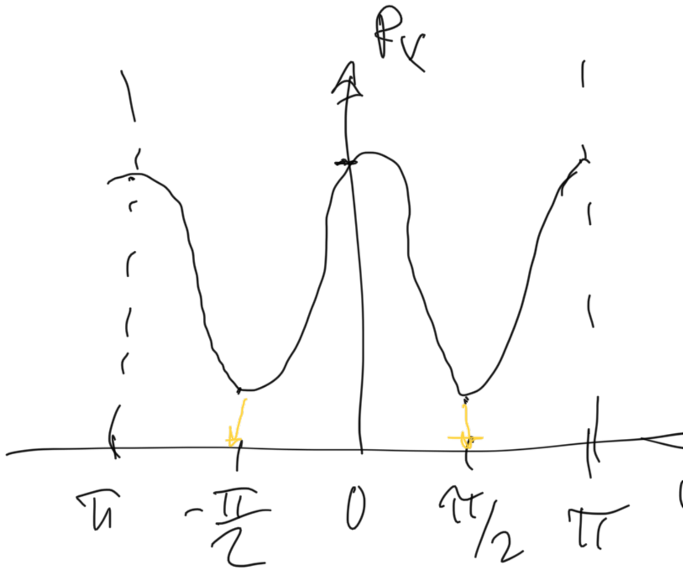






Ex) MA(2)





Conclusions:

AR: good for peaks in $P_x(e^{j\omega})$

MA: good for notches in $P_x(e^{j\omega})$

ARMA combines both!

Back to AR(1) process

$$x(n) + 0{,}5\,x(n-1) = e(n)$$

[Calc. R $r_x(k) = E\{x(n)\,x(n-k)\}$

... for $r_x(k) = E\{x(n)x(n-k)\}$

$= E\{x(n)x(n+k)\}$

$\Rightarrow r_x(k) = r_x(-k)$

symmetric?

$E\{x(n-1)(x(n)+0,5x(n-1))\} = x(n-1)e(n)$

$r_x(1) + 0,5\, r_x(0) = 0$

$x(n-2): \Rightarrow r_x(2) + 0,5\, r_x(1) = 0$

$\vdots$

Square the AR(1) relation

LHS:

$E\{(x(n)+0,5x(n-1))(x(n)+0,5x(n-1))\}$

$E\{x(n)(x(n)+0,5x(n-1))\} +$

$0,5\, E\{\underbrace{x(n-1)(x(n)+0,5x(n-1))}_{=0}\}$

$r_x(0) + 0,5\, r_x(1)$

RHS: $E\{e^2(n)\} = \sigma_e^2$

$$\Rightarrow \begin{cases} r_x(0) + 0,5\, r_x(1) = \sigma_e^2 \\ r_x(1) + 0,5\, r_x(0) = 0 \end{cases}$$

$$\begin{cases} \vdots \\ r_x(k) + 0{,}5\, r_y(k-1) = 0 \end{cases}$$

$$\boxed{r_x(k) = -0{,}5\, r_x(k-1)}$$

$|r_x(k)| > 0 \quad \forall k$

$\Rightarrow$ $r_x(k)$ has an infinite length if AR-process.

$$\begin{bmatrix} 1 & 0{,}5 \\ 0{,}5 & 1 \end{bmatrix} \begin{bmatrix} r_x(0) \\ r_x(1) \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} r_x(0) \\ r_x(1) \end{bmatrix} = \frac{1}{1-0{,}25} \begin{bmatrix} 1 & -0{,}5 \\ -0{,}5 & 1 \end{bmatrix} \Big[\,$$

Ex) $p = 2 \quad \sigma_e^2 = 1$

$$\begin{cases} r_x(0) + a_1 r_x(1) + a_2 r_x(2) = 1 \end{cases}$$

$$\begin{cases} r_x(1) + a_1 r_x(0) + a_2 r_x(1) = 0 \\ r_x(2) + a_1 r_x(1) + a_2 r_x(0) = 0 \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & a_1 & a_2 \\ a_1 & 1+a_2 & 0 \\ a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} r_x(0) \\ r_x(1) \\ r_x(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ d \end{bmatrix}$$

MA(1), $\sigma_e^2 = 1$

$$x(n) = \underline{e(n) - 0{,}5\, e(n-1)}$$

multiply $x(n-k)$ on bothside

$$\underbrace{E\{x(n)x(n-k)\}}_{r_x(k)} = E\{x(n-k)\left(e(n) - 0{,}5\, e(n-1)\right)\}$$

$$\left(e(n-k) - 0{,}5\, e(n-k-1)\right)$$

if $k > 1 \Rightarrow r_x(k) = 0$

$$r_x[0] = E\{e^2(n) + 0{,}5^2 e^2(n-1)\} = 1 + 0{,}25 = 1{,}25$$

$$r_x[1] = 0{,}5\, E\{e^2(n-1)\} = 0{,}5$$

for $k \geq 2$: $r_x[k] = 0$

For general MA(q) models.

$$x(n) = e(n) + b_1 e(n-1) + \dots + b_q e(n-q)$$

Def: $b(k) = \begin{cases} 1 & k = 0 \\ b_k & k = 1 \dots q \\ 0 & k > q,\ k < 0 \end{cases}$

$$r_x[k] = \sum_{n=0}^{q} b(n)\, b(k-n) \qquad -q \le k \le q$$

$\Rightarrow$ MA process has a finite length ACF

Estimate an AR(p) model from data. Given $\{x(n)\}_{n=0}^{N-1}$

estimate $a_1 \dots a_p$

1) Find $\hat{r}_x(k) = \frac{1}{N} \sum_{n=k}^{N-1} x(n)\, x(n-k)$

2) Use Y-W eqv. $k = 1 \dots p$

$$r_x(k) + a_1 r_x(k-1) + \dots a_p r_x(k-p) = 0$$

$$\text{Y-W} \Rightarrow \begin{bmatrix} r_x(0) & r_x(1) & \dots & r_x(p-1) \\ r_x(1) & r_x(0) & \dots & r_x(p-2) \\ \vdots & & \ddots & \vdots \\ r_x(p-1) & \dots & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(p) \end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{\underline{R}} \qquad \underline{a} \qquad \underline{r}$$

$$\hat{\underline{R}}\,\underline{a} = \hat{\underline{r}}$$

$$\hat{a} = \hat{\underline{R}}^{-1}\,\hat{\underline{r}}$$

$$\hat{a} = \hat{R} \backslash \hat{r} \quad \text{(in Matlab)}$$

$$\cancel{\mathrm{inv}(\hat{R}) * \hat{r}}$$