MVE 136 L13

ACF: $r_x(n) \triangleq E\{x(n_0)\,x(n_0-n)\}$: $r_x(n) = r_x(-n)$

PSD: $P_x(e^{j\omega}) \triangleq \mathrm{DTFT}\{r_x(n)\}$

Wiener - Khintchin Thm. alt. def.

$$P_x(e^{j\omega}) = \lim_{N\to\infty} \frac{1}{N} E\left\{ \left| X_N(e^{j\omega}) \right|^2 \right\}$$

when $X_N(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)\, e^{-j\omega n} = \mathrm{DTFT}\{x_N(n)\}$

$$x_N(n) = \begin{cases} x(n) & , \quad n = 0, \dots N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$\hat{P}_{per}(e^{j\omega}) \triangleq \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2$$

Strengths: 1) Efficient to compute FFT

2) $\lim_{N\to\infty} E\{\hat{P}_{per}(e^{j\omega})\} = P_x(e^{j\omega})$

$\therefore$ Periodogram is asymptotically unbiased.

An alternative route:

Since $P_x(e^{j\omega}) = \mathrm{DTFT}\{r_x(n)\}$

1) Estimate $r_x(n)$

$$\hat{r}_x(n) = \frac{1}{N} \sum_{n_0=n}^{N-1} x[n_0]\,x[n_0-n]$$

$n = 0, \pm 1, \pm 2, \dots \pm(N$

2) $\hat{P}_{per}(e^{j\omega}) = \mathrm{DTFT}\{\hat{r}_x(n)\}$

$\mathrm{DTFT}\{\hat{r}_x(n)\} = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2$

same result as before!

1) Finite sequence impact!
(finite freq resolution)

2) Expected value is correct.
How large is the var. of $\hat{P}_{per}(e^{j\omega})$

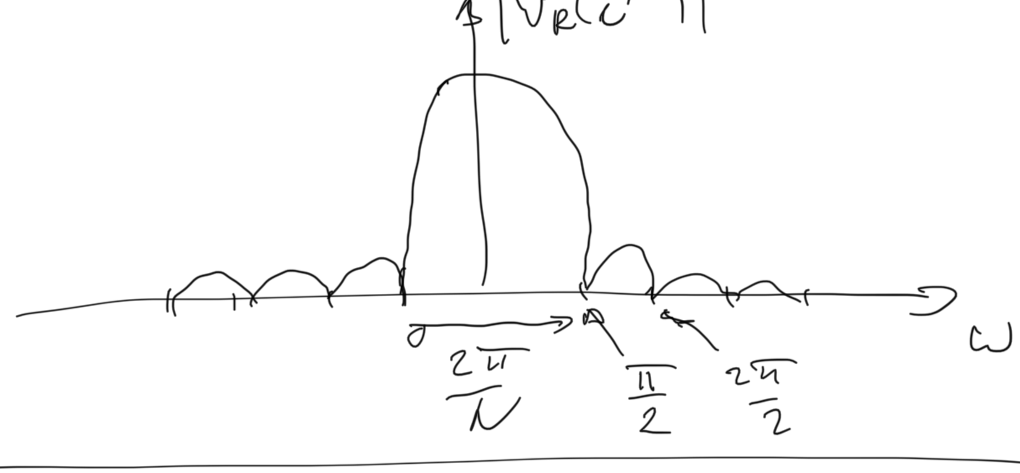
Bias for finite N?

$$\underline{w_R(n)} = \begin{cases} 1 & n = 0, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$X_N(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \underline{w_R(n)\, x(n)}\, e^{-j\omega n}$$

$$\Rightarrow \underline{X_N(e^{j\omega})} = W_R(e^{j\omega}) * X(e^{j\omega}) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} W_R(e^{jz})\, \underline{X(e^{j(\omega - z)})}\, dz$$



Variance of $\hat{P}_{per}(e^{j\omega})$

Assume large N $E\{\hat{P}_{per}(e^{j\omega})\} \approx P(e^{j\omega})$

Assuming large N ... $E\{\hat{P}_{per}\} \approx P_x(e^{j\omega})$

$$\Rightarrow \operatorname{var}\{\hat{P}_{per}(e^{j\omega})\} = E\left\{\left(\hat{P}_{per}(e^{j\omega}) - P_x(e^{j\omega})\right)^2\right\}$$

$$\approx P_x(e^{j\omega})$$

Note that $\operatorname{var}\{\hat{P}_{per}\} \not\to 0$ as $N \to \infty$

$\Rightarrow$ The estimate is not consistent.

$$e(n) \longrightarrow \boxed{AR} \longrightarrow x(n) \qquad x(0) \ldots x(N-1)$$

What is PSD

1) Estimate the AR-model $\to \dfrac{1}{\hat{A}(e^{j\omega})}$

2) $\hat{P}_x(e^{j\omega}) = P_{AR}(e^{j\omega}) = \left|\dfrac{1}{\hat{A}(e^{j\omega})}\right|^2 \sigma_e$

Parametric technique!

AR-estimate as LR.

$$x(n) = e(n) - a_1 x(n-1) - \cdots - a_p x(n-p)$$

$$\hat{x}(n) = -a_1 x(n-1) - \cdots - a_p x(n-p)$$

$$\min_{\{a_1 \ldots a_p\}} \sum_{n=p}^{N-1} \left(x(n) - \hat{x}(n)\right)^2 = \min_{\{a_i\}} L(a_1 \ldots a_p)$$

$$L(a_1 \ldots a_p) = \sum_{n=p}^{N-1} \left(x(n) + \sum_{i=1}^{p} a_i x(n-i)\right)^2$$

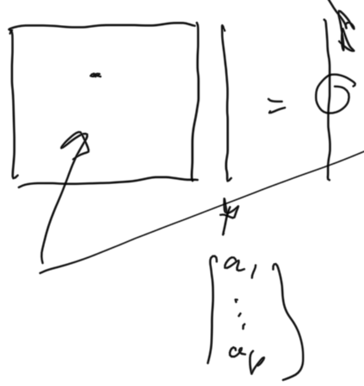
$$\frac{\partial}{\partial a_i} L(a_1 \ldots a_p) = 0 \qquad i = 1 \ldots p$$

$$\frac{\partial}{\partial a_i} L(\,) = \sum^{N-1} 2\left(x(n) + \sum_{i=1}^{p} a_i x(n-i)\right)\left(x(\ldots\right.$$

$n=p$

$$\Rightarrow \sum_{n=p}^{N-1} x(n)\,x(n-r) = -\sum_{n=p}^{N-1}\sum_{i=1}^{p} a_i\, x(n-i)\,x(\ldots$$

$r = 1 \ldots p$



p equations and p unknowns!

YW-equation with sample correlation!