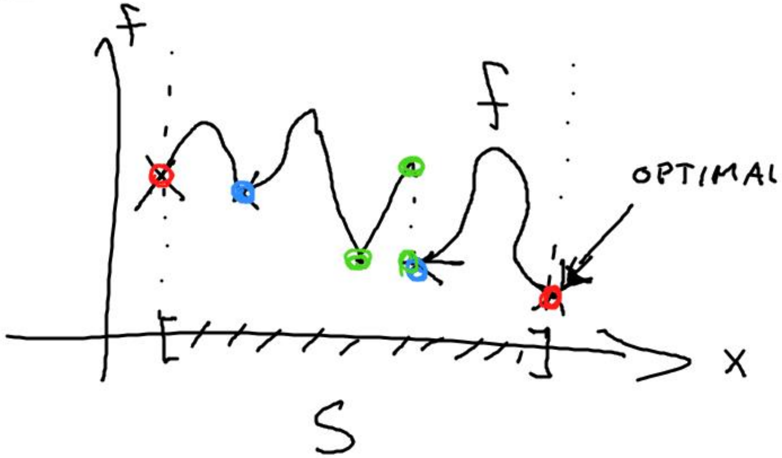
LECTURE 3 : INTRODUCTION TO OPTIMALITY CONDITIONS

$$\boxed{(P) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}}$$

S IS NONEMPTY $S \subseteq \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}$

WHEN $n=1$



WHERE CAN WE FIND OPTIMAL SOLUTIONS?

- BOUNDARY POINTS
- STATIONARY POINTS $f'(x)=0$
- DISCONTINUITIES OF f OR f'

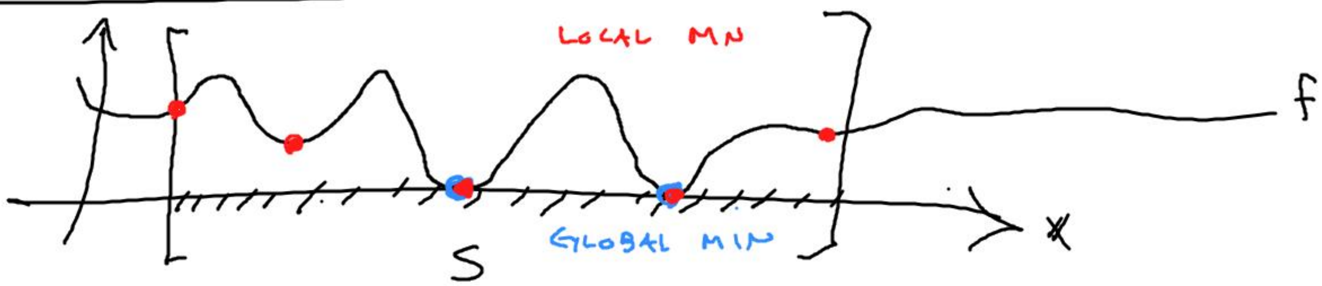
DEF: $x^* \in S$ is a GLOBAL MINIMUM of f over S if

$$f(x^*) \leq f(x) \quad \forall x \in S$$

DEF: $x^* \in S$ is a LOCAL MINIMUM of f over S if

$$\exists\, \varepsilon > 0 \text{ such that } f(x^*) \leq f(x) \quad \forall x \in S \cap B_\varepsilon(x^*)$$

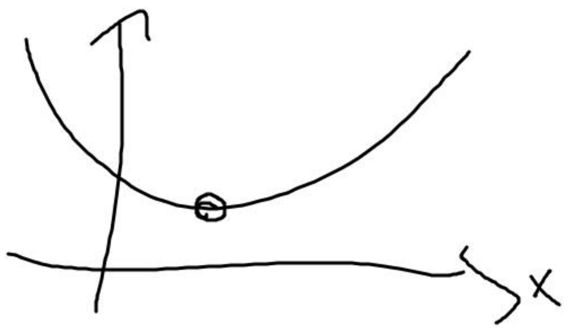
where $B_\varepsilon(x^*) = \{ y \in \mathbb{R}^n \mid \|x^* - y\| < \varepsilon \}$

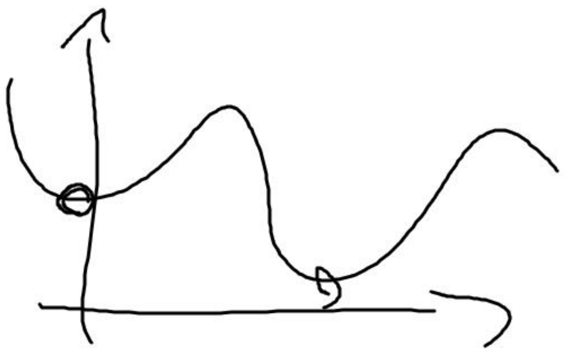


THM: (FUNDAMENTAL THM OF GLOBAL OPTIMALITY)

CONSIDER PROBLEM (P) WHERE S IS A CONVEX SET AND f IS CONVEX ON S. THEN EVERY LOCAL MINIMUM IS ALSO A GLOBAL MINIMUM

$$(P): \quad \min f(x) \quad \text{s.t. } x \in S$$





PROOF:

ASSUME $x^* \in S$ IS A LOCAL MINIMUM BUT NOT A GLOBAL MINIMUM. LET $\bar{x} \in S$ BE A POINT FOR WHICH

$$f(\bar{x}) < f(x^*) \qquad \text{(SINCE } x^* \text{ IS NOT GLOBAL MIN)}$$

TAKE $\lambda \in (0,1)$ $\qquad x^* \in S,\ \bar{x} \in S$

$$\lambda \bar{x} + (1-\lambda) x^* \in S \qquad \text{(SINCE } S \text{ IS CONVEX)}$$

$$\underline{f(\lambda \bar{x} + (1-\lambda) x^*)} \leq \lambda f(\bar{x}) + (1-\lambda) f(x^*) \qquad \text{(SINCE } f \text{ IS CONVEX)}$$

$$< \lambda f(x^*) + (1-\lambda) f(x^*) \underline{= f(x^*)} \qquad \text{(SINCE } f(\bar{x}) < f(x^*)\text{)}$$

LET λ BE SMALL ENOUGH

$\Rightarrow$ CONTRADICTS THAT x^* IS A LOCAL MIN.

$\Rightarrow$ ASSUMPTION NOT CORRECT $\blacksquare$

EXISTENCE OF OPTIMAL SOLUTIONS

Ex.

$\min\ e^{-x}$
s.t. $x \geq 0$

$$\boxed{(P)\quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array}}$$

NOTATIONS:

- $S \subseteq \mathbb{R}^n$ is OPEN if for every $x \in S$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) = \{ y \in \mathbb{R}^n \mid \|y - x\| < \varepsilon \} \subset S$
- $S \subseteq \mathbb{R}^n$ is CLOSED if $\mathbb{R}^n \setminus S$ is open
- $S \subseteq \mathbb{R}^n$ is BOUNDED if $\|x\| < M$ for all $x \in S$
- If a set S is closed and bounded it is COMPACT

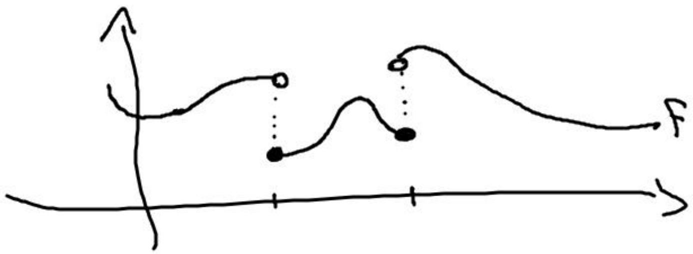
DEF: A FUNCTION f IS <u>WEAKLY COERCIVE</u> WITH RESPECT TO THE SET S IF EITHER S IS BOUNDET OR

$$\lim_{\substack{\|x\| \to \infty \\ x \in S}} f(x) = +\infty$$

e^{-x}

DEF: A FUNCTION f IS <u>LOWER SEMI-CONTINUOUS</u> AT x IF

$$x_k \to x \quad \Longrightarrow \quad f(x) \leq \liminf_{k \to \infty} f(x_k)$$



THM: (WEIERSTRASS' THOEREM).

CONSIDER PROBLEM (P) WHERE

- S IS NONEMPTY, CLOSED
- f IS LOWER-SEMI CONTINUOUS
- f IS WEAKLY COERCIVE W.R.T. S.

THEN THERE EXISTS A NONEMPTY, COMPACT SET OF OPTIMAL SOLUTIONS (GLOBAL MINIMA) TO (P)

$$(P) \quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in S \end{array}$$

PROOF: COURSE BOOK!

OPTIMALITY CONDITIONS WHEN $S = \mathbb{R}^n$

$$\boxed{(P) \quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in \mathbb{R}^n \end{array}}$$

NECESSARY CONDITIONS

x^* IS A LOCAL MIN $\Longrightarrow$ [SOMETHING HOLDS]

SUFFICIENT CONDITIONS

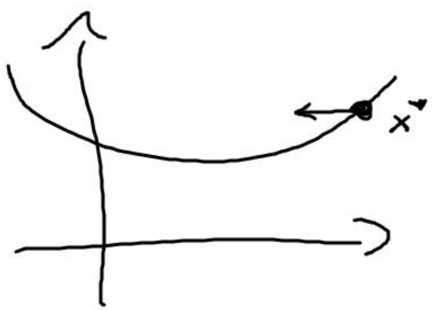
[SOMETHING HOLDS] $\Longrightarrow$ x^* IS A LOCAL MIN

NECESSARY CONDITION

THM: IF $f \in C^1$ ON $\mathbb{R}^n$, THEN

x^* LOCAL MIN OF f $\Longrightarrow$ $\nabla f(x^*) = 0$

$$(P) \quad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in \mathbb{R}^n \end{array}$$



PROOF: SUPPOSE x^* IS A LOCAL MIN BUT $\nabla f(x^*) \neq 0$

LET $p = -\nabla f(x^*)$

$$f(x^* + \alpha p) = f(x^*) + \alpha \nabla f(x^*)^T p + o(\alpha)$$

$$= f(x^*) - \alpha \underbrace{\|\nabla f(x^*)\|^2}_{>0} + o(\alpha)$$

FOR SMALL α, $o(\alpha) \to 0$

$$< f(x^*)$$

CONTRADICTS THAT x^* IS A LOCAL MIN. $\blacksquare$