LECTURE 6: OPTIMALITY CONDITIONS (KKT CONDITIONS)

RECAP:

$$\boxed{\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \leq 0 \quad i = 1, \dots, m \end{aligned}} \quad (P)$$

NECESSARY OPT CONDITION:

x^* LOCAL MIN TO (P) $\Rightarrow$ "THERE SHOULD NOT EXIST A DIRECTION FROM x^* THAT IS BOTH A FEASIBLE AND A DESCENT DIRECTION"

$$\left(\begin{aligned} &T_S(x^*) : \text{TANGENT CONE} \\ &\mathring{F}(x^*) = \{ p \in \mathbb{R}^n \mid \nabla f(x^*)^T p < 0 \} \end{aligned} \right) \Rightarrow \underbrace{A(x^*)}_{\text{"FEASIBLE DIRECTIONS"}} \cap \underbrace{B(x^*)}_{\text{"DESCENT DIRECTIONS"}} = \emptyset$$

$$\Rightarrow \boxed{T_S(x^*) \cap \mathring{F}(x^*) = \emptyset} \quad \text{GEOMETRIC OPT COND.}$$

$\Rightarrow$ FRITZ JOHN CONDITIONS.

<u>PROBLEM:</u> $T_S(x^*)$ IS MESSY TO COMPUTE!

$I(x^*)$ THE SET OF ACTIVE CONSTRAINTS

$$\mathring{G}(x^*) = \{ p \in \mathbb{R}^n \mid \nabla g_i(x^*)^T p < 0 , \ i \in I(x^*) \}$$ INNER GRADIENT CONE

$$G(x^*) = \{ p \in \mathbb{R}^n \mid \nabla g_i(x^*)^T p \leq 0 , \ i \in I(x^*) \}$$ GRADIENT CONE

$$\boxed{\mathring{G}(x) \subseteq T_S(x) \subseteq G(x)}$$

TOO SMALL TOO BIG

DEF: A CONSTRAINT QUALIFICATION (CQ) IS A REGULARITY CONDITION OF THE SET OPTIMIZING OVER

DEF: (ABADIE'S CQ)

ABADIE'S CQ HOLDS AT A POINT $x \in S$ IF $T_S(x) = G(x)$

WILL BE USED TO FORMULATE STRONGER OPT. CONDITIONS.

$$(P)\quad \boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad i=1,\dots,m \end{array}}$$

THM: (KARUSH – KUHN – TUCKER CONDITIONS)

ASSUME THAT ABADIE'S CQ HOLDS AT A POINT x^* WHICH IS FEASIBLE IN (P).

x^* LOCAL MIN $\Longrightarrow$ THE FOLLOWING SYSTEM IS SOLVABLE FOR μ:

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0, \quad i=1,\dots,m \\ \mu_i \geq 0 \end{cases}$$

PROOF: - USE FARKAS' LEMMA: $\begin{cases} A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0 \end{cases}$ $\begin{cases} A^T\mathbf{y} \leq 0 \\ \mathbf{b}^T\mathbf{y} > 0 \end{cases}$

- GEOMETRIC OPTIMALITY CONDITIONS:

$\mathbf{x}^*$ LOCAL MIN $\Rightarrow$ $T_S(\mathbf{x}^*) \cap \mathring{F}(\mathbf{x}^*) = \emptyset$

SINCE ABADIE'S CQ HOLDS

$\Rightarrow$ $G(\mathbf{x}^*) \cap \mathring{F}(\mathbf{x}^*) = \emptyset$

$\Rightarrow$ $\begin{cases} \nabla f(\mathbf{x}^*)^T \mathbf{p} < 0 & \text{(DESCENT DIRECTION)} \\ \nabla g_i(\mathbf{x}^*)^T \mathbf{p} \leq 0, & i \in I(\mathbf{x}^*) \text{ (FEASIBLE DIRECTION)} \end{cases}$ IS UNSOLUABLE

$\Rightarrow$ [FARKAS' LEMME]

LET A BE THE MATRIX WITH COLUMNS $\nabla g_i(x^*)$, $i \in I(x^*)$

THEN

$$\Longrightarrow \begin{cases} \nabla f(x^*)^T p < 0 \\ \nabla g_i(x^*)^T p \leq 0, \quad i \in I(x^*) \end{cases} \Longrightarrow \begin{cases} A^T p \leq 0 \\ -\nabla f(x^*)^T p > 0 \end{cases} \quad \text{UNSOLVABLE}$$

$$\Longrightarrow \text{[FARKAS' LEMMA]} \Longrightarrow \begin{cases} Az = -\nabla f(x^*) \\ z \geq 0 \end{cases} \quad \text{SOLVABLE}$$

DEFINE $\mu_{I(x^*)} = z$ AND $\mu_i = 0$ FOR ALL $i \notin I(x^*)$.

THEN THE VECTOR μ SATISFIES THE KKT CONDITIONS

(LINEAR INEQ SYSTEM) ∎

VERIFY

KKT CONDITIONS

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0 \quad , \; i=1,\dots,m \\ \mu_i \geq 0 \quad , \; i=1,\dots,m \end{cases} \Rightarrow$$

COMPLEMENTARY SLACKNESS

- IF $g_i(x^*)=0$, THE CONSTRAINT i IS ACTIVE

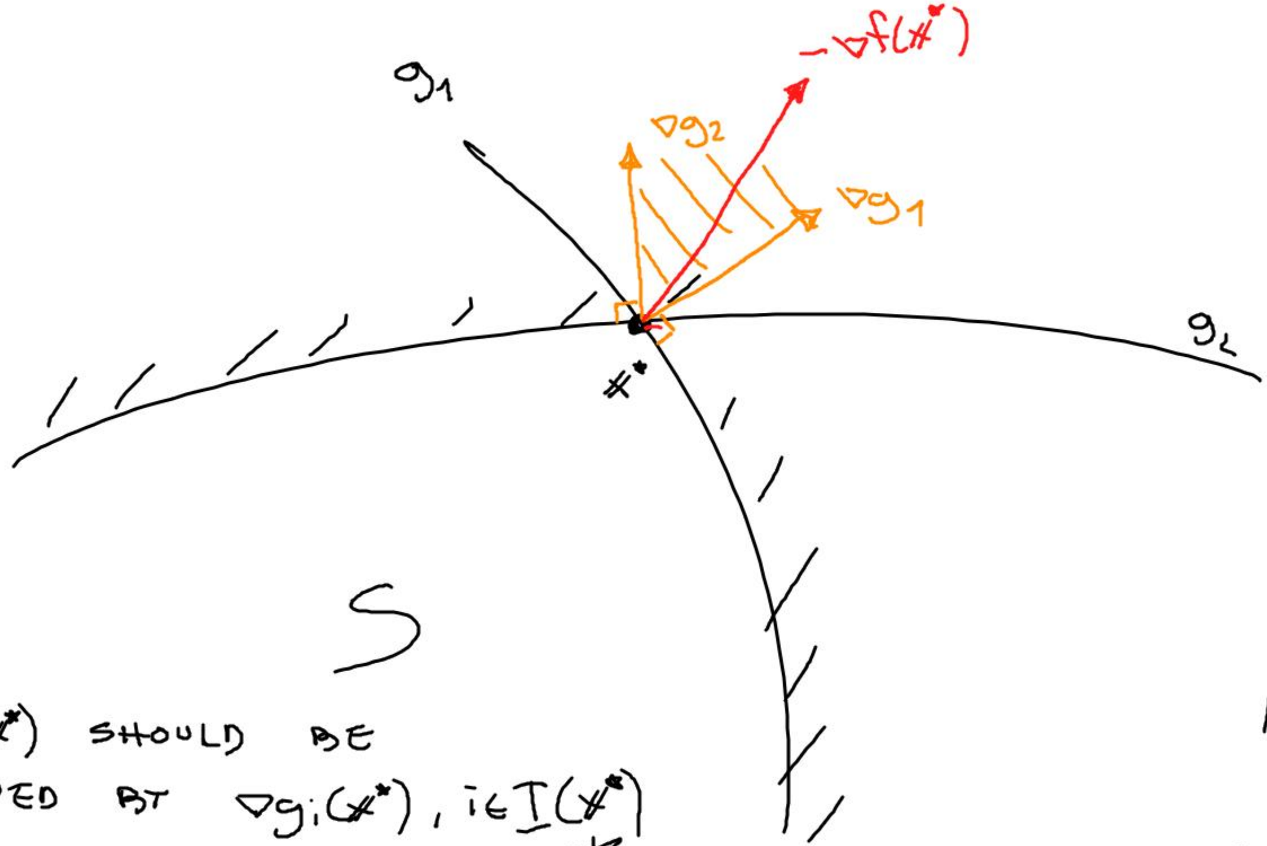
$$\Rightarrow \mu_i \geq 0$$

- IF $g_i(x^*)\neq 0$, THE CONSTRAINT i IS NOT ACTIVE

$$\Rightarrow \mu_i = 0$$

$$\Rightarrow \begin{cases} -\nabla f(x^*) = \sum_{i \in I(x^*)} \mu_i \nabla g_i(x^*) \\ \mu_i \geq 0 \; , \quad i \in I(x^*) \end{cases}$$

THE NEGATIVE GRADIENT SHOULD BE EXPRESSED AS A NON-NEGATIVE LINEAR COMBINATION OF THE GRADIENTS OF THE ACTIVE CONSTRAINTS



IF x^* IS A LOCAL MIN, THEN THE NEGATIVE GRADIENT $-\nabla f(x^*)$ SHOULD BE EXPRESSED AS A NON-NEGATIVE LIN. COMB. OF THE GRADIENTS OF THE ACTIVE CONSTRAINTS.

$-\nabla f(x^*)$ SHOULD BE SPANNED BY $\nabla g_i(x^*)$, $i \in I(x^*)$

REMARK: KKT CONDITIONS

$$\begin{cases} \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(x^*) = 0 \\ \mu_i g_i(x^*) = 0 \quad , i=1,\dots,m \\ \mu_i \geq 0 \quad , i=1,\dots,m \end{cases}$$

HOME ASSIGNMENTS:

- WHAT HAPPENS IF $m=0$ (NO CONSTRAINTS)?
- WHAT HAPPENS IF x^* IS AN INNER POINT? NO ACTIVE CONSTRAINTS. $I(x^*)=\emptyset$.

HOW CAN WE USE THEM?

- CAN ALWAYS CHECK CANDIDATE SOLUTIONS. $\bar{x}$
  IF KKT SYSTEM IS SOLVABLE, $\bar{x}$ MIGHT BE OPTIMAL
  IF KKT SYSTEM IS NOT SOLVABLE, $\bar{x}$ CAN NOT BE OPTIMAL, SHOULD SEARCH FURTHER.
- ONLY NEED TO SEARCH FOR OPTIMAL SOLUTIONS IN KKT POINTS
- CONSTRUCT METHODS THAT SEARCHES FOR KKT-POINTS

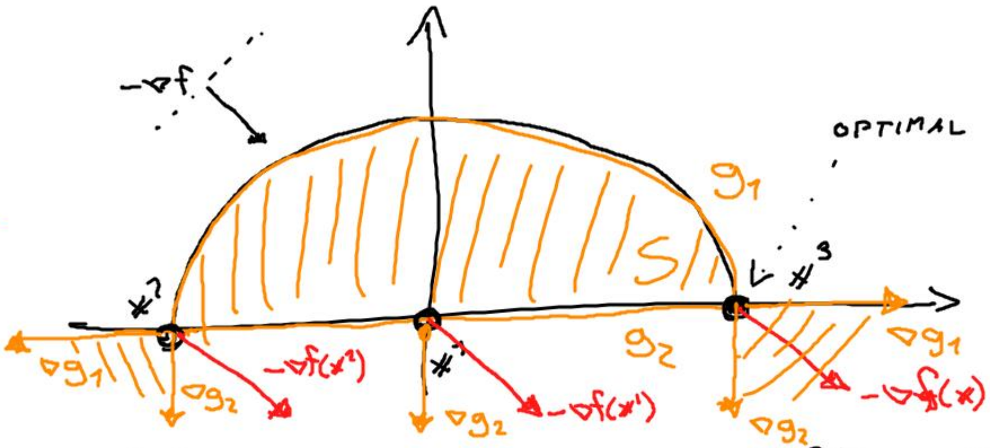
<u>REMARK:</u> A KKT-POINT IS A POINT WHERE THE KKT SYSTEM IS SOLVABLE

Ex.

$$\begin{cases} \min & -x_1 + x_2 \\ \text{s.t.} & x_1^2 + x_2^2 - 1 \le 0 \quad g_1 \\ & -x_2 \le 0 \quad g_2 \end{cases}$$

$$\nabla f(x) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \nabla g_1(x) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\nabla g_2(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$



$$x^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad I(x^1) = \{2\}$$

$$\nabla f(x) + \sum_{i \in I(x)} \mu_i \nabla g_i(x) = 0$$

$$\mu_i \geq 0$$

$$\begin{cases} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \mu_2 \geq 0 \end{cases}$$

$\boxed{\text{NOT KKT !}}$

$$x^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad I(x^2) = \{1, 2\}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu_1, \mu_2 \geq 0$$

$\boxed{\text{NOT KKT !}}$

$$x^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad I(x^3) = \{1, 2\}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mu_1, \mu_2 \geq 0$$

SOLVABLE

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = 1$$

$\boxed{\text{KKT POINT}}$

THM: KKT THEOREM

IF ABADIE'S CQ HOLDS AT x^*. THEN

x^* IS LOCAL MIN $\Longrightarrow$ x^* IS A KKT POINT

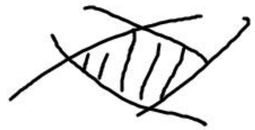
DEF: ABADIE'S CQ HOLDS AT x^* IF $T_S(x^*) = G(x^*)$

"SIMPLER CONSTRAIN QUALIF." $\Longrightarrow$ ABADIE'S CQ

DEF: (LICQ) LINEAR INDEPENDENCE CQ.

WE SAY THAT LICQ HOLDS AT x^* IF THE GRADIENTS $\nabla g_i(x^*)$, $i \in I(x^*)$ FOR THE ACTIVE CONSTRAINTS ARE LINEARLY INDEPENDENT

THM: LICQ $\Longrightarrow$ ABADIE'S CQ



DEF: (AFFINE CQ).

WE SAY THAT THE AFFINE CQ HOLDS IF ALL CONSTRAINT g_i ARE AFFINE (LINEAR

THM: AFFINE CQ $\Longrightarrow$ ABADIE'S CQ

DEF: SLATER'S CQ

WE SAY SLATER'S CQ HOLDS IF ALL g_i ARE CONVEX AND AN INNER POINT EXISTS. i.e. $\exists x_0 : g_i(x_0) < 0 \quad \forall i = 1, \dots, m$

THM: SLATERS' CQ $\Longrightarrow$ ABADIE'S CQ.

EMILS TIPS:

1: CHECK AFFINE CQ HOLDS

2: CHECK SLATER'S CQ

3: CHECK LICQ.

A NOTE ON EQUALITY CONSTRAINTS

$$\boxed{\begin{aligned} &\min\ f(x) \\ &\text{s.t.}\quad g_i(x) \le 0, \quad i=1,\dots,m \\ &\qquad\ \ h_j(x) = 0, \quad j=1,\dots,\ell \end{aligned}}$$

TRICK IS TO REFORMULATE

$$\boxed{\begin{aligned} &\min\ f(x) \\ &\text{s.t.}\quad g_i(x) \le 0, \quad i=1,\dots,m \\ &\qquad\ \ h_j(x) \le 0, \quad j=1,\dots,\ell \\ &\qquad -h_j(x) \le 0, \quad j=1,\dots,\ell \end{aligned}}$$

STATE KKT CONDITIONS FROM THIS AND USE SOME LOGIC.

SEE LECTURE NOTES AND BOOK

SUFFICIENCY OF KKT CONDITIONS

$$\min f(x)$$
$$\text{s.t. } g_i(x) \leq 0, \; i=1,\dots,m$$

THEOREM: IF THE OBJECTIVE FUNCTION f IS CONVEX AND ALL CONSTRAINT FUNCTIONS g_i ARE CONVEX, THEN

x^* KKT POINT $\Longrightarrow$ x^* GLOBAL OPTIMUM
(KKT SYSTEM IS SOLVABLE)

PROOF: SEE BOOK.