1. Canvas announcements
(Question)

2. Discussion forum

3. Information for
first day

"Are points in $\mathbb{Q} \cap [0,1]$ isolated?

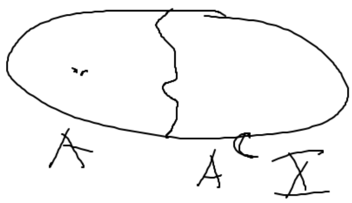
~~Dis~~ Totally disconnected.

1.3 Every σ-alg is finite or unctble

$(X, \mathcal{M})$ $|\mathcal{M}| < \infty$ or $|\mathcal{M}|$ unctble

$(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ $\mathcal{P}(\mathbb{N})$ unctble "Cantors diag."

Pf assume $|\mathcal{M}| = \infty$



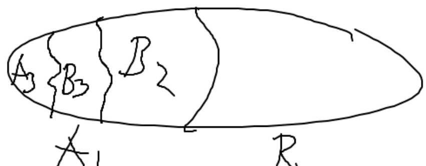
Choose A s.t $A \neq \phi, X$

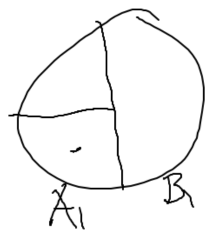
Call A infinite if the # of sets from $\mathcal{M}$ which are contain in A is inf. Either A or A^c infinite

Goal: $|M| = \infty$, $\exists$ inf # of disjoint sets

Let A_1 piece which is ∞ and B_1 infinite or finite

In same way,
A_1 can be broken
into 2 nonempty
pieces A_2, B_2
with A_2 infinite
Induction keep going





B_1
B_2
B_3
disjoint sets

Y Y N N Y N N ..

O O O O O O O ...

$B_1, B_2\ B_3\ B_4 \dots \longrightarrow B_1 \cup B_2 \cup B_5$

$NNYYYY \dots \longrightarrow B_3 \cup B_4 \cup B_5$ — | unctble # of such sequences

18a $\forall E \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists$ open set O $(\mathbb{R}, \mathfrak{M}, m)$

st $E \subseteq O$ and $m^*(O) \le m^*(E) + \varepsilon$

Pf case 1: $m^*(E) = \infty$ Take $O = \mathbb{R}$

case 2: $m^*(E) < \infty$ Choose $I_1, I_2 \cdots$ intervals st

$E \subseteq \bigcup I_i$, $\sum l(I_i) \le m^*(E) + \varepsilon$

let $O = \bigcup I_i$. O open $E \subseteq O$

$$\underline{m^*(O)} \cancel{\le} = m^*\left(\bigcup I_i\right) \le \sum_{i=1}^{\infty} \cancel{m}\, l(I_i) \le \underline{m^*(E) + \varepsilon}$$

5. let $m^*(E) < \infty$ (not msble maybe)

Then $\exists$ a ctble intersection of open sets $\bigcap_{i=1}^{\infty} O_i$

st $E \subseteq \bigcap_{i=1}^{\infty} O_i$ and $m^*(E) = m^*(\bigcap_{i=1}^{\infty} O_i)$ G_δ

Pf Choose (a) O_k st O_k open, $E \subseteq O_k$,

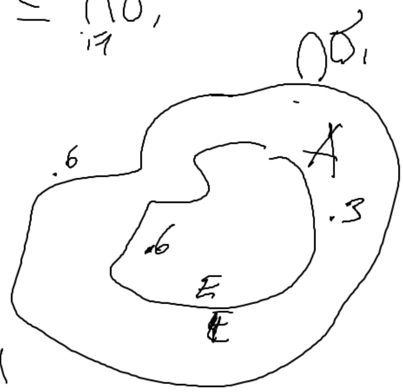
$m(O_k) \le m^*(E) + \boxed{1/k}\ \varepsilon_k \to 0$ Now $\bigcap_{i=1}^{\infty} O_i \supseteq E \Rightarrow m^*(E) \le m(\bigcap O_i)$

$m(\bigcap_{i=1}^{\infty} O_i) \le m(O_k) \le m^*(E) + 1/k$

$\forall k \Rightarrow m(\bigcap O_i) \le m^*(E)$

$\underline{m^*(E) < \infty}$

E is msble iff $\exists \bigcap_{i=1}^{\infty} O_i$ s t $E \subseteq \bigcap_{i=1}^{\infty} O_i$



and $m^*(\cap O_i \setminus E) = 0$

$\Leftarrow$ $m^*(A) = 0 \Rightarrow A \in \mathfrak{M} \Rightarrow$

$\underbrace{\cap O_i \setminus A}_{=E} \in \mathfrak{M}$

$\Rightarrow$ by 5', choose $\cap O_i$ s t $E \subseteq \cap O_i$ and

$m(\cap O_i) = m^*(E)$. E ~~sur~~ msble $\Rightarrow$

$\underline{m(\cap O_i)} = m^*(E) + m^*(A) = \underline{m(\cap O_i) + m^*(A)}$ $= 0$

msble E can be well approximated by nice sets,

like $\bigcap_{i=1}^{\infty} O_i$

E can be approx by compact sets from inside

Motivation for Exercises 30, 31

Obv. $I \subseteq A$ $\quad I = [a,b]$ $b>0$, $\Rightarrow m(A) > 0$

? If $m(A) > 0$, does A contain an interval ?

No $A = [0,1] \setminus \mathbb{Q}$

? If $m(A) > 0$. A closed $\Rightarrow$ ~~$m(A)$~~ $A \supseteq I$?

No Fat cantor sets

① Build the Cantor in the same way except remove sets of size $(1/4)^n$ at level n and not $(1/3)^n$

closed does not contain interval

$m(C_{1/4}) = 1 - \cancel{\sum_{n=1}^{\infty} \frac{2^n}{4^n}}$ ~~Exer~~

remove set of more < 1

$m < 1$

② $\mathbb{Q} \cap [0,1] = \{q_1, q_2, q_3, q_4 \ldots\}$

$$A = [0,1] \setminus \underbrace{\bigcup_{i=1}^{\infty} \left(q_i - \frac{1}{(10)2^i}, q_i + \frac{1}{(10)2^i}\right)}_{\text{open}}$$

A closed

$A \not\supset I$ since all $\mathbb{Q}$ removed

30 $m(E) > 0$ $\forall \alpha < 1$ $\exists$ I interval s.t. $m(E \cap I) > \alpha\, m(I)$

$E \supseteq J$ trivial

E fills α proportion

I

Pf Choose $I_1, I_2, \ldots$ s.t. $E \subseteq \bigcup I_i$

$\sum \ell(I_i) \le \frac{m(E)}{\alpha}$

$\underline{\sum_{i=1}^{\infty} m(I_i \cap E)} \underset{\text{sub.}}{\ge} m(E) \ge \underline{\alpha \sum \ell(I_i)}$

$\Rightarrow$ for some i, $|I_i \cap E| \ge \alpha |I_i|$.

31 $m(E)>0 \Rightarrow E-E \ (:=\{e_1-e_2 : e_1, e_2 \in E\}) \supseteq (-\varepsilon, \varepsilon)$ $\varepsilon>0$ some $\varepsilon>0$

Pf By (30), $\exists\, I$ st $m(E\cap I) \geq \frac{3}{4} m(I)$

$A := E\cap I \qquad B = (E\cap I) + x$

$A\cap B \neq \emptyset \Rightarrow x \in E-E$ Pf The pt in $A\cap B$ is $e_1 = e_2 + x$, $x = e_1 - e_2$

$A\cap B = \emptyset \quad m(A\cup B) = 2\,m(E\cap I) \geq \frac{3}{2} m(I)$

$A\cup B \subseteq I \cup (I+x) \Rightarrow m(A\cup B) \leq m(I\cup(I+x))$

$\leq \frac{3}{2} m(I) \Rightarrow A\cap B \neq \emptyset \Rightarrow x\in E-E$

$|x| \leq \frac{|I|}{2}$

$$\boxed{\begin{array}{l} |x| \leq \frac{|I|}{2} \Rightarrow x \in E-E \\ E-E \supseteq \left[-\frac{|I|}{2}, \frac{|I|}{2}\right] \end{array}}$$