Class Lectures (for Chapter 5)

Product measures

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The set is $X \times Y$. The σ-algebra , called $\mathcal{M} \times \mathcal{N}$, is $\sigma(\mathcal{R})$ where

$$\mathcal{R} := \{A \times B : A \in \mathcal{M}, B \in \mathcal{N}\}.$$

$\mathcal{R}$ stands for rectangles.

Existence of Product measures

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Theorem

(Existence of Product Measures) There exists a measure $\mu \times \nu$ on $(X \times Y, \mathcal{M} \times \mathcal{N})$ so that for all $A \times B \in \mathcal{R}$,

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B).$$

Moreover, $\mu \times \nu$ is the unique measure satisfying these properties if both μ and ν are σ-finite.

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Proof (read lecture notes)

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3. One proves $(\mu \times \nu)_0$ is a premeasure.
4. One uses our theorem on premeasures to obtain our measure.

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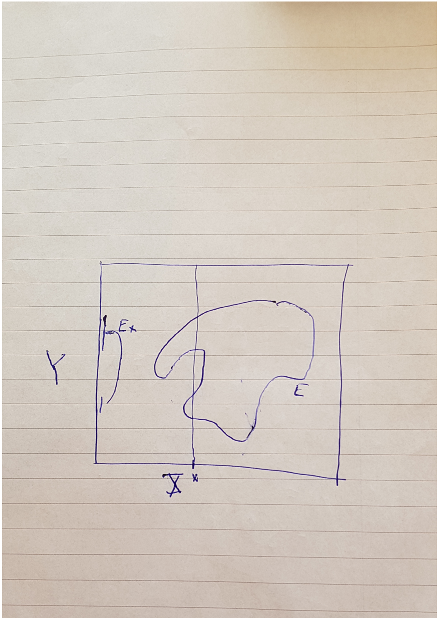

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$$E \in \mathcal{G} \to \forall x, E_x \in \mathcal{N} \to \forall x, (E_x)^c \in \mathcal{N} \to \forall x, (E^c)_x \in \mathcal{N} \to E^c \in \mathcal{G}.$$

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Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.
If $E \in \mathcal{M} \times \mathcal{N}$, then the maps

$$x \to \nu(E_x) \text{ and } y \to \mu(E^y) \text{ (well defined by the previous result)}$$

are measurable functions on X and Y respectively.

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We will do the proof in the finite case and the first part of each of the two lines.

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STEP 2: $\mathcal{C}$ is a $\mathcal{D}$-system.

Exercise.

Proof of Fubini's Theorem for Sets

Step 3. Using the fact that $\mathcal{R}$ is a π-system, Dynkin's $\pi - \lambda$ Theorem gives the second equality below and steps 1 and 2 give the containment.

$$\mathcal{M} \times \mathcal{N} = \sigma(\mathcal{R}) = \mathcal{D}(\mathcal{R}) \subseteq \mathcal{C}.$$

QED

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$$\int_X \nu(D_x) d\mu(x) = 1 \neq 0 = \int_Y \mu(D^y) d\nu(y).$$

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This is true but requires a little work. Details are in the lecture notes.

Tonelli's Theorem for Functions

Theorem

Let $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *be* σ*-finite measure spaces.*
If $f \in L^+(X \times Y)$*, then*

$$g(x) := \int_Y f_x(y) d\nu(y) \in L^+(X)$$

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- If f is the indicator function of some measurable set E, this is exactly Fubini's Theorem for finite sets.
- Since the same is true if we "first integrate with respect to x", the two "iterated integrals" are the same.

Fubini's Theorem for Functions

Theorem

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces.
If $f \in L^1(X \times Y)$, then

$$f_x \in L^1(Y, \mathcal{N}, \nu) \text{ for } \mu\text{-a.e. } x \in X.$$

Furthermore,

$$g(x) := \int_Y f_x(y) d\nu(y) \text{ (which is defined } \mu\text{-a.e.) belongs to } L^1(\mu)$$

and

$$\int_{X \times Y} f(x, y) d(\mu \times \nu)(x, y) = \int_X g(x) d\mu(x).$$

Fubini's Theorem for Functions

- Although the theorem is still true if we reverse x and y, it is possible that the two iterated integrals are finite and differ. (See Exercise 48, Chapter 2, Folland).

Fubini's Theorem for Functions

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- In real problems, one first takes absolute values, computes an iterated integral and uses Tonelli's Theorem to hopefully conclude that f is integrable in the product space. Then, one uses Fubini's Theorem to compute the integral.