P. 33 $X = \{0,1\}$

If we have the sequence $X_n \{\{0\},\{1\},\{0\},\{1\},\dots\}$
Then we have
lim sup $= \{0,1\}$ and lim inf $= \emptyset$.

- lim inf contains all subsets which are low bounds for all except finitly many of the sequence
- Analog for lim sup.

P. 41

$f:[0,1] \to \mathbb{R}$, let $C = \{x : f \text{ is cont. at } x\}$, which is Borel.

We will not only show Borel but also that it is G_δ set, i.e. a countable intersection of open sets, so

$$C = \underbrace{\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{\substack{I \text{ open} \\ \text{interval } \dim(f(I)) < \frac{1}{n}}} I}_{\text{open set}}}_{\text{countable intersection of open sets}}$$

$\therefore G_\delta$ and therefore Borel.

P. 44 $(f^{-1}(E))^c = f^{-1}(E^c)$

Proof: For any $x \in f^{-1}(E^c) \Leftrightarrow f(x) \in E^c \Leftrightarrow$
$f(x) \notin E \Leftrightarrow x \in (f^{-1}(E))^c$ $\square$

P.46 Exercise 1

We have that if $f: (X, \mathcal{M}) \to \overline{\mathbb{R}}$ then

$\{x : f(x) = \infty\} \in \mathcal{M}$ since

$$\{x : f(x) = \infty\} = \underbrace{\bigcap_{i=1}^{\infty} \underbrace{\{x : f(x) > n\}}_{\text{measurable}}}_{\substack{\text{countable intersection} \\ \text{of measurable sets}}}$$

P.46 Exercise 2

No, since if we let $f_n = 1 - 1/n$, then we have $\sup f_n = 1$, let $a = 1$, then

$$\underbrace{\{x : \sup f_n(x) \geq 1\}}_{= X} \neq \bigcup_n \underbrace{\{x : f_n(x) \geq 1\}}_{\emptyset}$$

P.50 Show $\int f+g \geq \int f + \int g$.

First we fix Φ simple such $0 \leq \Phi \leq f$ and ψ simple such $0 \leq \psi \leq g$

We have that $\Phi + \psi$ simple, $0 \leq \Phi + \psi \leq f+g$

$$\int (f+g) \geq \int (\Phi + \psi) = \int \Phi + \int \psi$$

$\boxed{\text{or sup at same time}}$

This is true $\forall \Phi$ as above (ψ fixed)

$\Rightarrow \int f+g \geq \int f + \int \psi$

This is true $\forall \psi$ as before $\Rightarrow$

$\int f+g \geq \int f + \int g$.

$\leq$ Does not hold.

Take n simple $0 \leq n \leq f+g$ and its enough to show $\int n \leq \int f + \int g$

We need to decompose $n = \Phi + \psi$ with $0 \leq \Phi \leq f$, $0 \leq \psi \leq g$, then

$\int n = \int \Phi + \int \psi \leq \int f + \int g \to$ Issue!

P.56 Exercise 1

Take $([0,\infty], \mathcal{B}, m)$, Let $f_n = \frac{1}{n} I_{[0,n]}$

Certenly $f_n \to 0$ uniformly and $\int f_n = 1 \not\to 0$.

Comment: By LDC, $\nexists g \in L^1$ s.t $|f_n| \leq g$ $\forall n$.

This can't be done on a finite measure space
Since constants are integrable on finite measure space
and then $g = 1$ can dominate f_n, Which we don't
want

Exercise 2

No converse to LDC, $([0,\infty], \mathcal{B}, m)$

For example $f_n = \frac{1}{n^2} I_{[n^2, n^2+n]}$

$f_n \to 0$ uniformly, and $\int f_n = \frac{n}{n^2} = \frac{1}{n} \to 0$

$\nexists g \geq |f_n|$ $\forall n$, With $g \in L^1$ (Integrable)

assume $g \geq |f_n|$ $\forall n$

$$\int g \geq \sum_{n=1}^{\infty} \underbrace{\int_{n^2}^{n^2+n}}_{\text{disjoint for different } n} g \geq \sum_{n=1}^{\infty} \int_{n^2}^{n^2+n} f_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Exercise 3

Take $([0,1], \mathcal{B}, m)$, $f_n = n I_{[1/n+1, \frac{1}{n}]}$

Then $f_n(x) \to 0$ $\forall x$ (not uniform in x)

$$\int f_n = n\left(1/n - \frac{1}{n+1}\right) = \frac{1}{n+1} \to 0$$

No dominating function. If we assume $g \geq |f_n|$ $\forall n$

$$\int g = \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} g \geq \sum_{n=1}^{\infty} \underbrace{\int_{1/n+1}^{1/n} f_n}_{=\int f_n} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$$