Class Lectures (for Chapter 6)

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So, $(\Omega, \mathcal{M}, P)$ governs some "random experiment" where $P$ tells us the "likelihood" that $\omega$ (chosen "randomly") falls in different sets.

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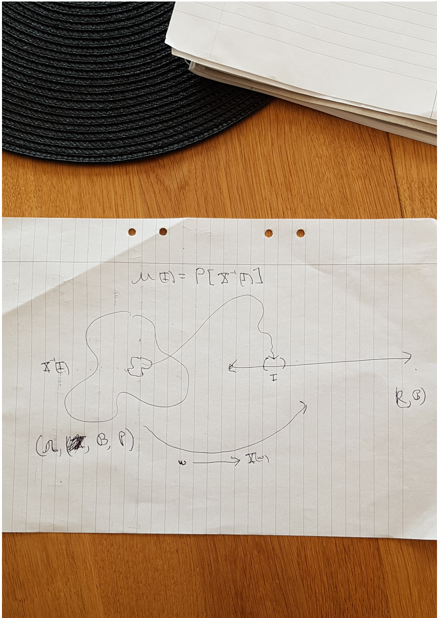
If $X$ is a random variable on a probability space $(\Omega, \mathcal{M}, P)$, its **expectation**, denoted $E(X)$, is simply defined by

$$E(X) = \int X dP$$

provided this exists, meaning at least one of $\int X^+ dP$ and $\int X^- dP$ is finite.

# Law or distribution of a random variable



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$$\mu_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

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An infinite collection of random variables on a probability space $(\Omega, \mathcal{M}, P)$ is called **independent** if each finite collection is independent as above.

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QED For the unit interval with Lebesgue measure, let $E_n = [0, 1/n]$, what is happening?

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Why? $P(|I_{E_n} - 0| \geq \epsilon) = P(E_n)$ which goes to 0. Hence convergence in measure. However, the second Borel-Cantelli Lemma says that $P(\limsup E_i) = 1$ and this means that it is not the case that $I_{E_n}$ converges to 0 a.s.. In fact it says that $I_{E_n}$ converges to 0 only on a set of probability 0.

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Hence $P(S_n/n \to 0) = 1$.

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# General SLLN

**Theorem**

*(Strong Law of Large Numbers: General case) Let* $X_1, X_2, \ldots$ *be independent random variables with the same distribution with* $E(|X|) < \infty$*. Then*

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What is happening? How could this be occuring?

For very large $n$, $\frac{S_n}{n}$ is very likely to be close to 0, but if you watch the trajectory in time, there will be these very rare times at which $\frac{S_n}{n}$ is close to $\infty$ and times close to $-\infty$.

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One can show that $X_1, X_2, \ldots$ are independent and each has distribution $(\delta_1 + \delta_{-1})/2$.

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*More specifically, one has that a.e.,* $\limsup_{n\to\infty} \sum_{k=1}^{n} \frac{X_k}{k^\alpha} = \infty$ *and* $\liminf_{n\to\infty} \sum_{k=1}^{n} \frac{X_k}{k^\alpha} = -\infty$*.*

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Explanation: The variance of $\sum_{k=1}^{n} \frac{X_k}{k^\alpha} = \sum_{k=1}^{n} \frac{1}{k^{2\alpha}}$ converges to $\infty$ if and only if $\theta \leq 1/2$.

# A few words about the variance

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- Assuming $X$ has finite expectation, $Var(X) < \infty$ if and only if $X \in L^2(\Omega, \mathcal{M}, P)$
- $Var(X) = E(X^2) - (E(X))^2$, which is something you might have seen, is actually the pythagorean theorem, viewed properly.

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Hence $E(X)$ and $X - E(X)$ are orthogonal. The pythagorean theorem tells us that $E(X^2) = E(X - E(X))^2 + (E(X))^2 = Var(X) + (E(X))^2$. So, the variance is the "squared distance from $X$ to its projection onto the 1-dimensional space of constant functions".

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Since $\frac{S_n}{n}$ approaches 0 in probability (WLLN),
$P(\frac{S_n}{n} < -.001) \leq P(|\frac{S_n}{n} - 0| \geq .001)$ which goes to 0 as $n \to \infty$.

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False:

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False: the above limit is not zero and rather equals

$$\frac{2}{\pi} \arcsin(\sqrt{.1})$$

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What's happening? The WLLN is **not** applicable since the $Y_i$'s are not independent. In fact, they are very correlated.