19 b. Find $f_n \to f$ unif, $f_n \in L^1$ $\int f_n \not\to \int f$ $\mu(X) = \infty$

$([0,\infty), \mathcal{B}, m)$ $f_n = \frac{1}{n} I_{[0,n]}$ $f = 0$ $\int f_n = 1 \ \forall n$

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38 (a) $f_n \to f$ $g_n \to g$ both in measure

$f_n + g_n \to f+g$ in measure Fix $\varepsilon > 0$

$\mu(x : |f_n + g_n - (f+g)| \geq \varepsilon) \leq \mu\{x : |f_n(x) - f(x)| + |g_n(x) - g(x)| \geq \varepsilon)$

$\leq \mu(\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\} \cup \{x : |g_n(x) - g(x)| \geq \varepsilon/2\}$

$\leq \underbrace{\mu(x : |f_n - f(x)| \geq \varepsilon/2)}_{\to 0 \text{ as } n \to \infty} + \underbrace{\mu(x : |g_n - g| \geq \varepsilon/2)}_{\to 0 \text{ as } n \to \infty}$

$\to 0$ as $n \to \infty$ $\quad$ `` $\quad$ $\to 0$

Lecture notes 57, 59, 64, 71

57 $\{x : f_n(x) \text{ converges}\} \in \mathcal{M}$. Seen given f_n, f $\{x : f_n(x) \to f(x)\} \in \mathcal{M}$

$$\bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x : |f_n(x) - f(x)| \le 1/m\}$$

On $\mathbb{R}$ a sequence a_n converges iff (a_n) is Cauchy

Def (a_n) is Cauchy if $\forall \varepsilon > 0, \exists N$ st $\forall n, m \ge N$ $|a_n - a_m| < \varepsilon$.

$$\bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \bigcap_{m=k}^{\infty} \{x : |f_n(x) - f_m(x)| < 1/m\}$$

$\{f_n\}$ msble $\qquad \overline{\lim_n}\, f_n(x)$ msble $\qquad \underline{\lim}\, f_n(x)$ msbl

$\{x : f_n(x) \text{ converges}\} = \{x : \overline{\lim} f_n(x) = \underline{\lim}\, f_n(x)\} = \{x : \overline{\lim} f_n - \underline{\lim}\, f_n = 0\}$

§9 (F 34(a)) LDC holds even if $f_n \to f$ in measure

Thm $f_n \to f$ in msure $\exists g \in L^1(X, \mathcal{M}, \mu)$ s.t $|f_n| \le g$ $\forall n$

Then $\int f_n \to \int f$

Pf If $\int f_n \not\to \int f$, $\exists$ a subsequence n_k st $\int f_{n_k} \xrightarrow{*} L \ne \int f$

Still have $\{f_{n_k}\} \to f$ in msre. By earlier result, $\exists$ further subsequence $\{n'_k\}$ st $f_{n'_k} \to f$ a.e

Apply LDC to $f_{n'_k}$ and get $\int f_{n'_k} \to \int f$

$* \Rightarrow \int f_{n'_k} \to L$

$\downarrow$

64. $E \subseteq X \times Y$ $\quad E_x = \{y : (x,y) \in E\}$

$f: X \times Y \to \mathbb{R}$, $f_x : Y \to \mathbb{R}$ defined $f_x(y) = f(x,y)$

$f = I_E$ Then $\forall x$, $f_x = I_{E_x}$

$$f_x(y) = f(x,y) = I_E(x,y) = \begin{cases} 1 & y \in E_x \\ 0 & y \notin E_x \end{cases} = I_{E_x}(y)$$

7) X was a random var on $(\Omega, \mathcal{F}, P)$

Defined the dist or law of X, μ_X as a p.m on $(\mathbb{R}, \mathcal{B})$

$\mu_X(A) = P(X^{-1}(A))$. Claim μ_X p.m

$\mu_X(\emptyset) = P(X^{-1}(\emptyset)) = P(\emptyset) = 0$. NTS $A_1, A_2, \dots$ disjoint

$\boxed{\mu_X(\mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1}$

in $\mathcal{B}$, then $\mu_X(\cup A_i) = \sum \mu_X(A_i)$

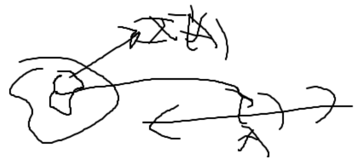
(1) $\forall$ $X^{-1}(A_1), X^{-1}(A_2)$, — disjoint $w \in X^{-1}(A_1) \cap X^{-1}(A_2) \Rightarrow X(w) \in A_1 \cap A_2 = \emptyset$

(2) $X^{-1}(\bigcup_{i=1}^{\infty} A_i) = \cancel{X^{-1}} \bigcup_{i=1}^{\infty} X^{-1}(A_i)$ $\supseteq$ $\subseteq$ $w \in LHS$ $X(w) \in \bigcup_{i=1}^{\infty} A_i \Rightarrow$

$X(w) \in A_i$ some i

$w \in X^{-1}(A_i)$ some i

$\Rightarrow w \in \bigcup X^{-1}(A_i)$



$\mu_X(\bigcup_i A_i) = P(X^{-1}(\bigcup A_i)) \overset{2}{=} P(\bigcup_i X^{-1}(A_i)) \overset{\&1}{=} \sum_i P(X^{-1}(A_i)) = \sum_i \mu_X(A_i)$ $\square$

3. ✓ 16 $f \in L^+$ $\int f < \infty$. $\forall \varepsilon > 0$ find E with $\mu(E) < \infty$

and $\int_E f > \int f - \varepsilon$ (trivial if $\mu(X) < \infty$)

Fix $\varepsilon > 0$

$f_n = f \, I_{\{x:\, f(x) \ge 1/n\}}$ f

$f_n(x) \nearrow f(x)$ $\underset{\substack{\text{MCT} \\ \text{LDC}}}{\Longrightarrow}$ $\int f_n \nearrow \int f$ choose N s.t. $\int f_N > \int f - \varepsilon$

$\int_{\{x:\, f(x) \ge 1/N\}} f > \int f - \varepsilon$ → finite msre?

Yes since o.w. $\int f = \infty$ ↯

$f_n \to f$, $g_n \to g$ in measure $\quad f_n g_n \longrightarrow fg$ in measure ?

Yes, if $\mu(X) < \infty$. No if $\mu(X) = \infty$

Example of No when $\mu(X) = \infty$ $\quad ([0,\infty), \mathcal{B}, m)$

$f_n \equiv 1/n \quad f \equiv 0 \quad g_n(x) = x \quad g(x) = x \quad f_n \to f \quad g_n \to g$

$f_n g_n(x) = \underline{\frac{x}{n}} \qquad fg = \underline{0}$

$\frac{x}{n} \not\to 0$ in measure ~~since~~ since

$m(\{x : |\frac{x}{n} - 0| \geq 1\}) = m([n,\infty)) = \infty \quad \forall n$

$f_n \to f$ $\quad g_n \to g \Rightarrow f_n g_n \to fg$, if $\mu(X) < \infty$ $\quad$ Fix $\varepsilon > 0$

$\mu(|f_n g_n - fg| \geq \varepsilon) = \mu(|f_n g_n - \underline{f g_n} + \underline{f g_n} - fg| \geq \varepsilon)$

$\leq \mu(|f_n g_n - f g_n| + |f g_n - fg| \geq \varepsilon)$

$\leq \mu(\{x : |f_n g_n - f g_n| \geq \varepsilon/2\} \cup \{|f g_n - fg| \geq \varepsilon/2\})$

$\leq \mu\{\{x : |f_n g_n - f g_n| \geq \varepsilon/2\} + \underline{\mu(|f g_n - fg| \geq \varepsilon/2)}$

Need show each term $\to 0$

$\mu\left(|f|\,|\cancel{f}g_n - \cancel{f}g| \geq \frac{\varepsilon}{2}\right)$ Fix $\delta > 0$ $\quad \varepsilon, \delta, N$

Choose N so that $\mu(\{x : |f(x)| \geq N\}) < \delta$

why? $\bigcap_{n=1}^{\infty} \{x : |f(x)| \geq n\} = \emptyset \Rightarrow \mu(x : |f(x)| \geq n) \xrightarrow[n\to\infty]{} 0$

const $\mu(A) < \infty$

$\leq \mu\left(\{x : |f| \geq N\} \cup \{x : |g_n(x) - g(x)| \geq \frac{\varepsilon}{2N}\}\right)$

If $|f| \leq N$, then $|Ng_n - Ng| \geq \varepsilon/2$

$\leq \delta + \mu\{x : |g_n(x) - g(x)| \geq \varepsilon/2N\} \to 0$ as $n \to \infty$

$\overline{\lim}\, \mu(|fg_n - fg| \geq \varepsilon/2) \leq \delta + 0$ True $\forall \delta \Rightarrow$

$\lim \mu(|fg_n - fg| \geq \frac{\varepsilon}{2}) \to 0$

$\mu(|f_n g_n - f g_n| \geq \varepsilon/2) \to$

Lemma: $\forall \delta > 0, \exists M$ s.t $\underline{\forall n}$ $\mu(|g_n| \geq \underline{M}) < \delta$

x $\boxed{\forall \delta > 0, \forall n, \exists M \quad \text{s t} \quad \mu(|g_n| \geq M) < \delta \qquad \text{did}}$

x $\boxed{g - g_n \quad \forall \delta > 0, \exists M \text{ s.t } \forall i = 1 \dots n \ \mu(|g_i| \geq M) < \delta \text{ easy}}$

choose M' s t $\mu(|g| \geq M') < \frac{\delta}{2}$ $\qquad \varepsilon = 1$ in conv in mea

$\exists N_0$ s t $n \geq N_0$ $\mu(|g_n| \geq M' + 1) \leq \underline{\mu(|g_n - g| \geq 1)} + \mu(|g| \geq M')$

$\leq \frac{\delta}{2} + \frac{\delta}{2} \leq \delta$. & For $i = 1, \dots N_0 - 1$, choose M_i s t

$\mu(|g_i| \geq M_i) \leq \delta$. Let $M = \max\{M_1, M_2, \dots M_{N_0 - 1}, M' + 1\}$

Now $\to 0$ using Lemma + previous page