Discussion pts from Last time

LDC, ν, μ $\quad \nu = \nu_s + \nu_{ac}$ with $\nu_s \perp \mu$ and $\nu_{ac} \ll \mu$

also $\nu_s \perp \nu_{ac}$. Why? Easy $\rightarrow$ $\exists A$ st $\nu_s(A^c) = 0 = \mu(A)$

$\underset{\nu_{ac} \ll \mu}{\Rightarrow}$ $\nu_{ac}(A) = 0$ $\quad \Rightarrow \nu_s \perp \nu_{ac}$

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full decomp on $(\mathbb{R}, \mathcal{B})$ $\quad m =$ Lebesgue

any $\nu = \underline{\nu_d} + \underline{\nu_{sc}} + \nu_{ac}$

$\nu_d(Q^c) = 0$ $\quad$ atomic $\quad$ cont (no atoms) $+ \nu_{sc} \perp m$ $\quad \ll m$

$Q = \{x : \nu_d(\{x\}) > 0\}$

$\underline{\nu_d}$, $\underline{\nu_{sc}}$, $\underline{\nu_{ac}}$

mutually sing

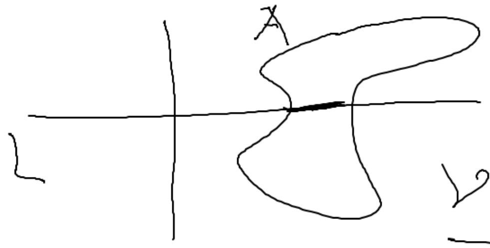
True in $\mathbb{R}^n$ $\underline{(\mathbb{R}^2, \mathcal{B})}$ m_2 = 2-dim Leb msre. $(m \times m)$

If ν is a σ-finite msre on ____,

$$\boxed{\nu = \nu_A + \nu_{sc} + \nu_{ac}}$$

we've seen cont sing msre on $\mathbb{R}$. Cantor msre μ_C,

This is much easier in $\mathbb{R}^2$. "1-d Leb msre restricted to x-axis"



$\nu(A) = m(A \cap L)$ ✓ ν is a σ-finite msre

ν has no atoms. ν cont

$\nu(\mathbb{R}^2 \setminus L) = 0$ $m_2(L) = 0$

$\underline{\nu \perp m_2}$

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1. $|X| < \infty$, $m = P(X)$, describe when $\nu \perp \mu$

$\nu = (\nu_1, \nu_2, \dots \nu_n)$ $\mu = (\mu_1, \dots \mu_n)$

$\nu \perp \mu \iff$ $\boxed{\mu_{support} := \{i : \mu_i > 0\} \quad \nu_{support} := \{i\ \nu_i > 0\}}$

$\mu_{supp} \cap \nu_{support} = \emptyset$

$\Rightarrow$ by contradi, $\exists i : \nu_i \mu_i > 0$

$\nu \perp \mu \Rightarrow \exists A$ s t $\mu(A) = 0$ $\nu(A^c) = 0$

case (1) $i \in A \Rightarrow$ $0 = \mu(A) \geq \mu_i > 0$ ↯

case (2) $i \notin A$ $0 = \nu(A^c) \geq \nu_i > 0$ ↯

$\Leftarrow$ Let $A := \{i \ \nu_i > 0\}$

$\nu(A^c) = \sum_{i \in A^c} \nu_i = \sum_{i \in A^c} 0$

$\mu(A) = 0$? $(\Rightarrow \nu \perp \mu)$.

$\mu(A) = \sum_{i \in A} \mu_i$ ~~ic~~ ~~is~~ $= 0$

$i \in A \Rightarrow \nu_i > 0 \Rightarrow \mu_i = 0$

1 0 3 0 0

0 2 0 0 4

$\{1,3\}$ $\{2,5,4\}$

2. atomic msre (ctbl # of atoms) $\perp \mu_c \perp m$

any such atomic msre is $\perp$ cont measure ν

$\mu_{d}(\underbrace{\mathcal{C}^c}_{\text{atomic}}) = 0 \qquad \underbrace{\nu}_{\text{cont}}(\underbrace{\mathcal{C}}_{\text{ctble}}) = 0$

$\underline{\mu_c \perp m} \qquad m(\underbrace{C}_{\text{cantor set}}) = 0 = \underline{\mu_c}(C^c)$

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$\delta_x \perp \delta_y$ $x \neq y$ (✓) $\{\delta_x\}_{x \in [0,1]}$ unctble # of mut sing

find unctble # of continuous mutually sing msres on $[0,1]$

$\forall p \in (0,1)$, construct a RV X_p taking values in $[0,1]$

We will show that the law of X_p, μ_p, are cont $\forall p$,

and $\mu_p \perp \mu_{p'}$ $p \neq p'$

Let Y_1^p, Y_2^p, Y_3^p --- indep and $P(Y_i^p=1)=p$

$P(Y_i^p=0)=1-p$

Law of $\underline{Y_i^p}$ is $\underline{p\delta_1+(1-p)\delta_0}$

Let $X_p = \sum_{i=1}^{\infty} \frac{Y_i^p}{2^i}$ $(Y_1^p, Y_2^p, \dots)$ terms in the binary expansion of X_p

$\{0,1\}^{\mathbb{N}} \longrightarrow [0,1]$

$(a_1, a_2, a_3, \dots) \longrightarrow \sum_{i=1}^{\infty} \frac{a_i}{2^i}$

almost bij.

onto

there are a ctble # of pts in $[0,1]$ with >1 preimage

$.0011111111\dots$ $.9999$

$=.01000000$ 1.0000

μ_p cont

$x = a_1(x), a_2(x), \dots a_n(x) \dots$

$\mu_p(\{x\}) = P[X_p = x] = P[\sum_{i=1}^{\infty} Y_i^p/2^i = x]$

$x \in [0,1)$

$= P(Y_1^p = a_1(x), Y_2^p = a_2(x), \dots Y_n^p = a_n(x))$

$\overset{\text{indep}}{=} (p \text{ or } (1-p)) \times (p \text{ or } (1-p)) \times \dots = 0$

$p \in (0,1)$

$p = 1 \quad X_1 \equiv 1$

$\mu_1 = \mu_{X_1} = \delta_1$

$\forall p$ SLLN $P\left[\frac{\sum_{i=1}^{n} Y_i^p}{n} \longrightarrow p\right]$ as

$A_p \subseteq [0,1] := \{x \in [0$

$$\frac{\sum^{n} Y_i^p}{n} \longrightarrow p \quad \text{a.s}$$

$A_p \subseteq [0,1]$: $\{ x \in [0,1] : x$ has a binary expansion $a_1^x, a_2^x, \ldots$ with

$$\frac{\sum_{i=1}^{n} a_i(x)}{n} \to p \}$$

$A_p \cap A_{p'} = \emptyset$ $p \neq p'$ (since $\frac{\sum_{i=1}^{n} a_i(x)}{n}$ converges to ≤ 1 limit)

$$\mu_p(A_p) = P[X_p \in A_p] = P\left[\sum_{i=1}^{\infty} \frac{Y_i^p}{2^i} \in A_p\right]$$

$$= P\left(\frac{\sum_{i=1}^{n} Y_i^p}{n} \to p\right) = 1$$

$$\boxed{\begin{aligned} * \ & A_p \cap A_{p'} = \emptyset \\ & \mu_p(A_p) = 1 \ \forall p \end{aligned}} \Rightarrow \mu_p \perp \mu_{p'}$$

$$\mu_p(A_p^c) = 0 \quad \mu_{p'}(A_p) \underset{*}{\le} \mu_{p'}(A_{p'}^c) = 0 \quad \square$$

$$\boxed{\mu_{1/2} = m}$$

It is much easier to find an unctble collection of continuous measures which are mutually sing in $\mathbb{R}^2$

Let $L_y = \{(x,y) : x \in \mathbb{R}\}$ $\quad \{L_y\text{'s}\}$ unct #

cont ~~$L_y \cap L_{y'}$~~ $\quad L_y \perp L_{y'} \quad y \neq y'$ Exercise $\square$