PS 86 $n = |X| < \infty$

$\nu \ll \mu$ $(\nu_1, \ldots \nu_n) \ll (\mu_1 \ldots \mu_n)$ iff $(\mu_i = 0 \Rightarrow \nu_i = 0)$

$(1,0,0,1) \ll (1,0,2,1/2)$ $(1\,2\,0\,1) \not\ll (1,0,2,1/2)$

RN Derivative of $\nu$ wrt $\mu$ $\left(\frac{d\nu}{d\mu}\right)$

$$f(i) = \begin{cases} \frac{\nu_i}{\mu_i} & \mu_i > 0 \\ \text{anything} & \mu_i = 0 \end{cases}$$

$\nu = f\,\cancel{d\mu}\,\mu$

$$\nu(\{i\}) = \int_{\{i\}} f\, d\mu \Rightarrow \nu_i = f(i)\,\mu_i$$

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$\nu = \underline{(1,0,1,0)}$ $\mu = (0,1,1,0)$

$\cancel{(1,0,1,0)}$

$\underline{(1,0,0,0)} + \underline{(0,0,1,0)}$

$\perp\mu$ $\ll\mu$

$\nu, \mu$ $A = \{ i \; \underline{\nu_i > 0}, \mu_i = 0 \}$

$\nu = \nu|_A + \nu|_{A^c} \to \ll \mu$

$\nu|_A(B) = \nu(A \cap B)$ $\perp\mu$

$\nu|_A(A^c) = 0$

$\mu(A) = 0$

$\mu_i = 0 \Rightarrow$ $\boxed{2}|_A$

2 cases

(1) $i \in A$ ✓ (2) $i \notin A$

LD of $\nu$ wrt $m$

$\nu = \underbrace{(2/3)\,\underline{\delta_{-7}} + 100\,\underline{\delta_4} + 3\,\underline{\delta_{5.6}}}_{\nu_s \perp m} + \underbrace{x^2 m|_{[4,6]}}_{\nu_{ac} \ll m}$  $\boxed{m = dx}$

$f\,m$

$= \int_A f\,d$

$\nu_s\left(\{-7, 4, 5, 6\}^c\right) = 0$

$\cancel{m}\left(\{-7, 4, 5, 6\}\right) = 0$

$\nu_{ac} \ll m|_{[4,6]} \ll m$

What RN deriv of $\nu_{ac}$ wrt $m$

$f(x) = x^2\, I_{[4,6]}$ ✓

$$\nu = \frac{2}{3}$$

$$\frac{2}{3}\delta_{-7} + 100\,\delta_4 + 3\,\delta_{5.5} + x^2 m|_{[4,6]} \qquad \nu$$

$$\frac{1}{4}\delta_4 + \frac{1}{2}\delta_{5.5} + \delta_{5.6} + 3x^3 m|_{[5,7)} \qquad \mu$$

$$\underbrace{\frac{2}{3}\delta_{-7} + x^2 m|_{[4,5)}}_{\nu_s \perp \mu} + \underbrace{100\,\delta_4 + 3\,\delta_{5.5} + x^2 m|_{[5,6]}}_{\nu_{ac} \ll \mu}$$

$$\frac{d\nu_{ac}}{d\mu} = \begin{cases} 400 & x = \cancel{400}\ 4 \\ 6 & x = 5.5 \\ 0 & x = 5.6 \\ (1/3x) & x \in [5,6] \setminus \{5.5, 5.6\} \\ 0 & \text{otherwise} \end{cases}$$

How does one get the $1/3x$? Hint

Prop 3.9 pg 91 Folland

not necess L.M

$\nu \ll \mu \quad \frac{d\nu}{d\mu} = g \qquad \mu \ll m \quad \frac{d\mu}{dm} = f$

$\nu \ll m \quad ( m(A) = 0 \Rightarrow \mu(A) = 0 \Rightarrow \nu(A) = 0 )$

$\frac{d\nu}{dm} \text{"="} \frac{d\nu}{d\mu} \frac{d\mu}{dm} = gf \quad (\Rightarrow \; 1/3x \;)$

=

Pf "usual way". First prove it for $g = I_A$

M ... with + M( ... $\forall a$ ...

Hint:

$\nu \ll \mu \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \nu(A) < \varepsilon$

$\underline{\nu(X) < \infty}$ $\quad \Leftarrow$ always

---

Folland pg 92 (2nd part) Above false if $\nu(X) = \infty$

$\nu =$ counting measure $\nu(i) = 1 \ \forall i$ $\quad \mu(i) = \frac{1}{2^i}$

$X = \{1, 2, 3, \dots\}$ $\quad \nu \ll \mu$

---

Take $\varepsilon = \frac{1}{2}$ $\quad$ claim no $\delta$

$\mu(i) = \frac{1}{2^i} < \delta \Longrightarrow \nu(i) < \varepsilon = \frac{1}{2}$ $\quad \downarrow$ $\nu(i) = 1$

Pg 9 2 13

RN Thm + LDC fail in non $\sigma$-finite m

$X = [0,1]$ $m$, $\underbrace{c, m}_{\text{not } \sigma\text{-finite}}$

---

$m \ll cm$ $\nexists f$ s.t. $m = f(cm)$. assume $\exists f$

$A = \{x\}$ $\underset{0}{\underset{\shortparallel}{m\{x\}}} = f(x) \cdot 1 \implies f \equiv 0$

$\underset{1}{\underset{\shortparallel}{m[0,1]}} \neq \underset{0}{\underset{\shortparallel}{\int_X f \, d(cm)}}$

c.m has no Leb decomp wrt m

$\boxed{\text{m does have a LD wrt c.m} \quad m = 0 + \underset{\ll c.m}{m}}$

$c\,m = \underbrace{\nu_s}_{\perp m} + \underbrace{\nu_{ac}}_{\ll m}$

$m(X) = 0$

$\Rightarrow \nu_{ac}\{X\} = 0$

$\cancel{\nu_s\{X\} + \nu_{ac}\{X\} = m\{X\})}$ $\Rightarrow \nu_s\{X\} = 1$

$\nu_s(A) = 0$ $\qquad m(A^c) = 0$

$\Downarrow$ $\qquad m([0,1]) = 0$ ↯

$A = \emptyset$