Example of approx of f by cont g

$f = I_{\mathbb{Q}}$ on $[0,1]$. $\quad g = 0 \quad$ ( $f = g$ ae )

$\int |f-g|\,dx = 0$

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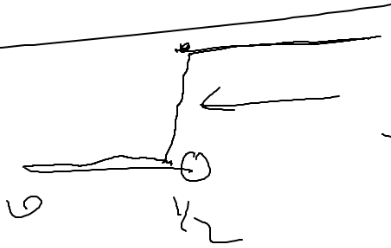
$f = I_{[1/2,1)}$



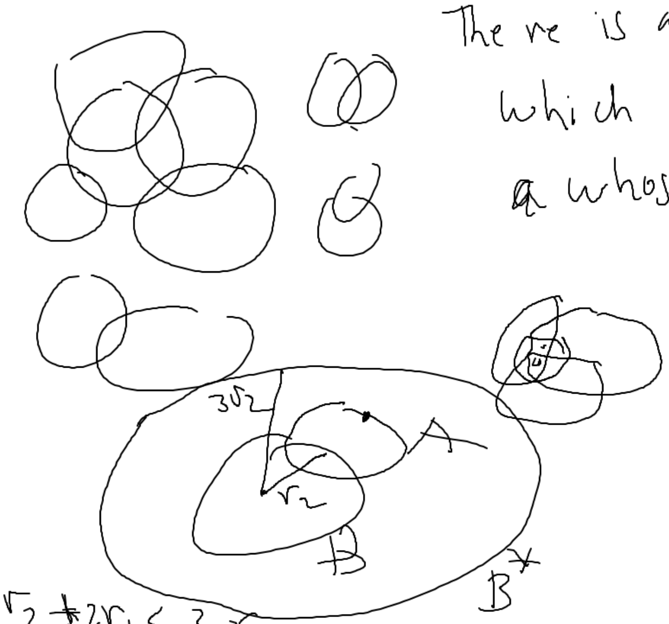

g $\int |f-g| < \varepsilon$

Let ~~$B_1$~~ $A_1, A_2, \ldots A_k$ be $k$ disks in $\mathbb{R}^2$ $\longrightarrow B_1, \ldots, B_\ell$

There is a ~~subset~~ subcollection of ~~the~~ these disks,

which are disjoint,

whose area is $\geq \frac{1}{9}$ original area




$r_2 + 2r_1 \leq 3r_2$

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$B$ ball $B^*$ has same center as $B$ but 3 times the radius $A(B^*) = \underset{3^2}{9}\,A(B)$

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$A, B$ balls $A \cap B \neq \emptyset$

$\text{rad } B \geq \text{rad } A$

$A \subseteq B^*$

$A_1, \ldots A_\ell$

$B_1$ be the largest ball. Remove all $A_i$'s which intersect $B_1$

$B_2$ - - - - - among the remaining ones Remove balls which intersect $B_2$

$B_3$ - - - - - - - - - - - - $B_3$

Keep going until the end, $B_1, \ldots B_\ell$ disjoint

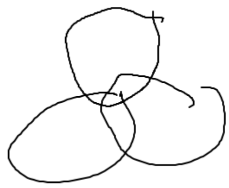
$\forall i, \exists j$ $A_i \subseteq B_j^*$. If $A_i$ is one of $B_1 \ldots B_\ell$, trivial

assume $A_i$ not any of $B_1 \ldots B_\ell$. So $A_i$ was removed at some stage. Let's say removed at stage 3. Then $A_i \cap B_3 \neq \emptyset$


$\text{rad}(A_i) \leq \text{rad}(B_3)$ since OW chosen $A_i$ instead of $B_3$

$A_i \subseteq B_3^*$

$$\frac{1}{9}\, m\Big(\bigcup_{i=1}^{k} A_i\Big) \le m\Big(\bigcup_{j=1}^{\ell} B_j^*\Big) \le \sum_{j=1}^{\ell} m(B_j^*)$$



$$= 9 \sum_{j=1}^{\ell} m(B_j) = \cancel{9}\, m\Big(\bigcup_{j=1}^{\ell} B_j\Big)$$

disjoint

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works in $\mathbb{R}^n$ $\qquad m(B^*) = 3^n m(B)$

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$A_1 \ldots A_k$ $\exists$ disjoint $B_1 \ldots B_\ell$

$$m\Big(\bigcup B_i\Big) \ge \frac{1}{3^n}\, m\Big(\bigcup_{i=1}^{\ell} A_i\Big)$$