MVE 460 F. 1.2

Adams Bok

§ 1.3 - 1.4

Mål:
1) Gränsvärden vid oändligheten (sidan 73)
2) oändliga gränsvärden
3) Teknik för rationella funktioner $\frac{p(x)}{q(x)}$
4) Kontinuitet i en punkt
5) Kontinuitet på ett intervall

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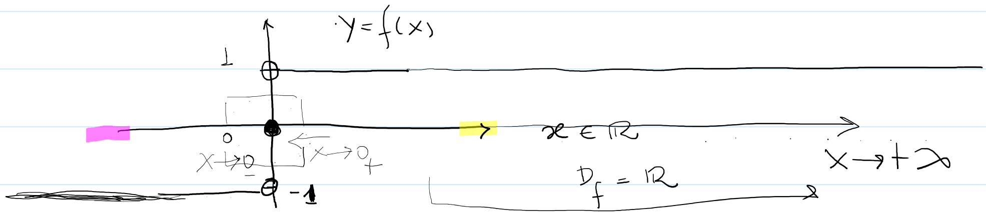
1) Gränsvärden vid oändligheten: $\lim\limits_{x\to+\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$

Exempel

1.1 $f(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$

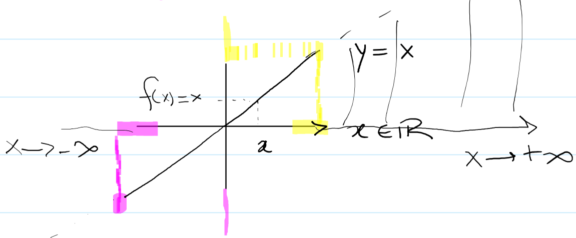


$\lim\limits_{x\to+\infty} f(x) = 1$ $\lim\limits_{x\to-\infty} f(x) = -1$

$p(x) = 1$ $\partial p(x) = 0$

$p(x) = 1x^0$

Ex 1.2 $f(x) = x$



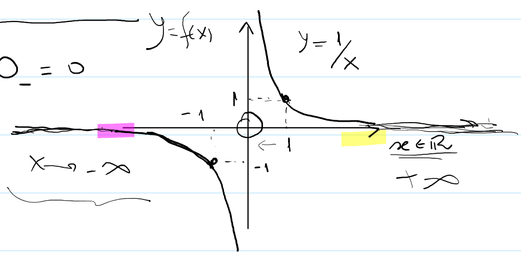
$f(x) = 1x^1$ $\partial p(x) = 1$

$\lim\limits_{x\to+\infty} f(x) = +\infty$ $\lim\limits_{x\to-\infty} f(x) = -\infty$

Ex 1.3

$f(x) = \frac{1}{x}$, $x \neq 0$ $D_f = \mathbb{R}\setminus\{0\} = (-\infty,0)\cup(0,+\infty)$

$\lim\limits_{x\to-\infty} \frac{1}{x} = 0_- = 0$



$\lim\limits_{x\to+\infty} \frac{1}{x} = 0_+ = 0$

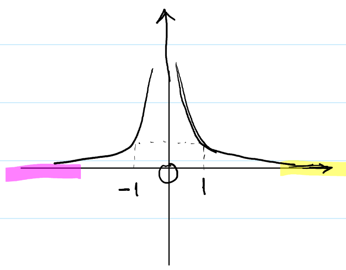
x	$f(x) = 1/x$
1	1
10	1/10

x	$f(x) = 1/x$
-1	$\frac{1}{-1} = -1$
-10	$-1/10 = -0{,}1$

10	1/10	−10	−1/10 = −0,1
100	0,01	−100	−0,01
10^4	0,0001	-10^{-4}	−0,0001
	↓ 0_+		↓ 0_-

Ex 1.4

$f(x) = \dfrac{1}{x^2}$



$x \in \mathbb{R}$

$D_f = \mathbb{R} \setminus \{0\}$

$V_f = \mathbb{R}_+ \setminus \{0\} = \mathbb{R}_+^*$

$$\lim_{x\to+\infty}\left(\frac{1}{x^2}\right) = \frac{\lim\limits_{x\to+\infty} 1}{\lim\limits_{x\to+\infty} x^2} = \frac{1}{+\infty} = 0_+ = 0$$

$$\lim_{x\to-\infty}\frac{1}{x^2} = \frac{1}{+\infty} = 0_+ = 0$$

Reglarna $\lim\limits_{x\to a} f(x) \pm g(x) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

Ex 1.5 Hitta $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+x} - x\right)$

1st) Enligt Reglerna kan man göra:

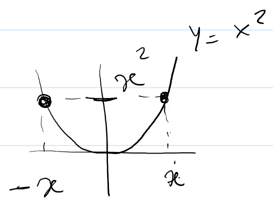
$$\lim_{x\to\infty}\left(\sqrt{x^2+x}\right) - \lim_{x\to+\infty} x = \infty - \infty \;???$$

Det finns inge slutsats här.

2a) Så måste man Rationalisera

$$f(x) = \left(\sqrt{x^2+x} - x\right) = \left(\sqrt{x^2+x} - x\right)\frac{\left(\sqrt{x^2+x} + x\right)}{\left(\sqrt{x^2+x} + x\right)}$$

$$= \frac{\cancel{x^2}+x + \cancel{x\sqrt{x^2+x}} - \cancel{x\sqrt{x^2+x}} - \cancel{x^2}}{\sqrt{x^2+x} + x} =$$



$$= \frac{x}{\sqrt{x^2+x} + x} = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)} + x} =$$

$$\sqrt{x^2\left(1+\frac{1}{x}\right)} = \sqrt{x^2}\sqrt{1+\frac{1}{x}}, \quad \sqrt{x^2} = |x|$$

$$= \frac{x}{|x|\left(\sqrt{1+\frac{1}{x}}\right) + x}$$

Obs: $\underbrace{\sqrt{x^2}} = |x| = \begin{cases} \boxed{x, \text{ om } x>0} \\ -x, \text{ om } x<0 \end{cases}$ Men vi vill $x \to +\infty$

Då $\underline{x>0}$

Så $\lim\limits_{\boxed{x\to+\infty}} f(x) = \lim\limits_{x\to+\infty} \dfrac{x}{x\left(\sqrt{1+\frac{1}{x}}\right)+x} =$

$+\infty$

$= \lim\limits_{x\to+\infty} \dfrac{\cancel{x}}{\cancel{x}\left[\left(\sqrt{1+\frac{1}{x}}\right)+1\right]} =$

$= \lim\limits_{x\to+\infty} \dfrac{1}{\left(\sqrt{1+\frac{1}{x}}\right)+1} = \dfrac{1}{1+1} = \dfrac{1}{2}$ ✓

$\frac{1}{x} \to 0_+$

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2. Oändliga gränsvärden

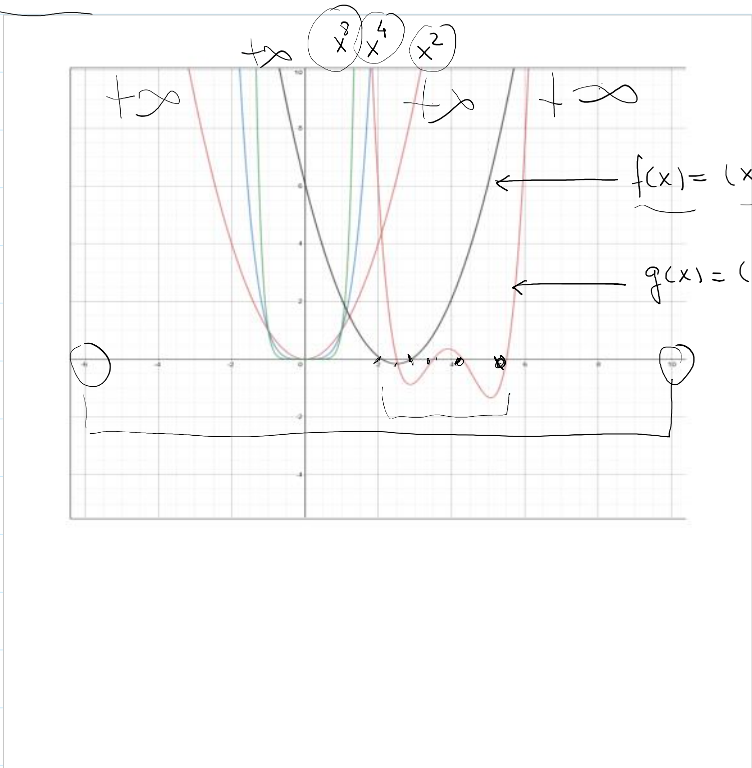
Obs: $p(x) = a_n x^{n} + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

$a_n \neq 0$, $\partial p = n$

$a_i \in \mathbb{R}$, $i = 0, 1, \dots n$

Om $n = 2, 4, 6, \dots 2k \equiv$ jämnt tal och $\boxed{a_n > 0}$

$a_n < 0$ $f(x) = -x^2$



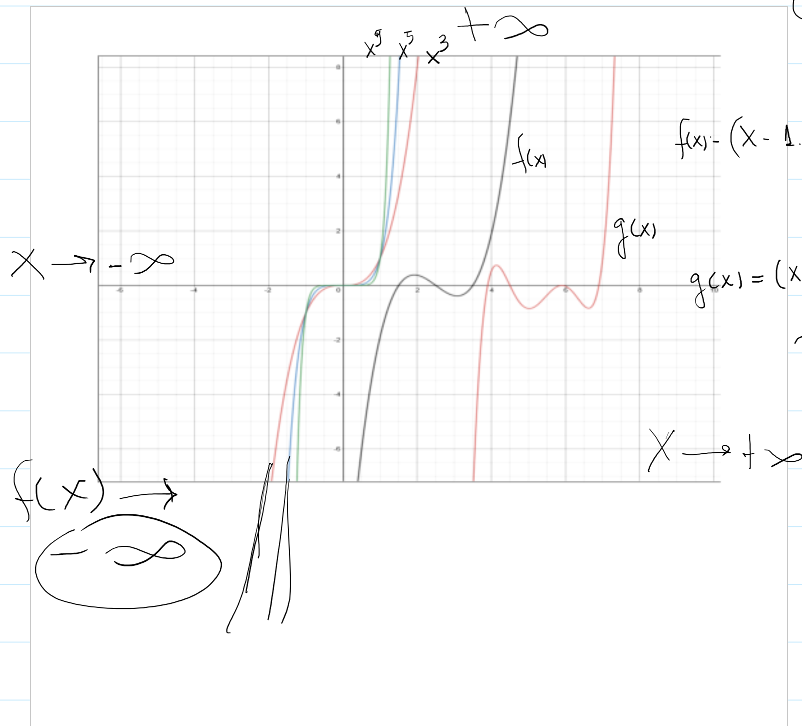
$f(x) = (x-2,1)(x-2,9)$

$g(x) = (x-2,5)(x-3,5)(x-4,3)(x-5,5)$

Om $n = 1, 3, 5, 7 \dots, \underbrace{2k+1} \equiv$ udda tal & $a_n > 0$

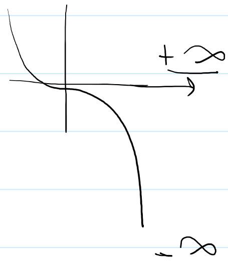
$a_n > 0$

$f(x) = -x^3$



$a_n > 0$

$f(x) = -x$



$f(x) \to -\infty$

$f(x) = (x-1.5)(x-2.5)(x-3.5)$

$\partial f(x) = 3$

$g(x) = (x-3.9)^1(x-4.5)^1(x-5.9)^2(x-6.8)^1$

$\partial g(x) = 5$

3. Teknik för Rationella funktioner

3.1. $f(x) = \dfrac{p(x)}{q(x)}$ $q(x) \neq 0$

Om p(x) och q(x) är polynomial, då kan man simplifiera!

Ex: (a) $\lim\limits_{x \to +\infty} \dfrac{\boxed{2}x^2 - x + 3}{\boxed{3}x^2 + 5}$

$p(x) = a_2 x^2 + a_1 x + a_0 \leftarrow$

$q(x) = b_2 x^2 + b_1 x + b_0 \leftarrow$

$\partial p = \partial q = 2$

(a) $\partial p(x) = \partial q(x) = (\text{Här} = 2)$

$$f(x) = \frac{\cancel{x^2}\left(2 - \frac{1}{x} + \frac{3}{x^2}\right)}{\cancel{x^2}\left(3 + \frac{5}{x^2}\right)} = \frac{\left(2 - \frac{1}{x} + \frac{3}{x^2}\right)}{\left(3 + \frac{5}{x^2}\right)}$$

Då: $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{\left(2 - \frac{1}{x} + \frac{3}{x^2}\right)}{\left(3 + \frac{5}{x^2}\right)} = \dfrac{2}{3} = \boxed{\dfrac{a_2}{b_2}}$

($\frac{1}{x} \to 0$, $\frac{3}{x^2} \to 0$, $\frac{5}{x^2} \to 0$)

$\partial p = \partial q \Rightarrow \lim\limits_{x \to \infty} \dfrac{p(x)}{q(x)} = \dfrac{a_n}{b_n} \leftarrow$

$$\partial p = \partial q \Rightarrow \lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a_n}{b_n} \leftarrow$$

(b) $\boxed{\partial p(x) < \partial q(x)}$

$$\lim_{x\to+\infty} \frac{5x+2}{2x^3-1} =$$

$$\lim_{x\to+\infty} \frac{x(5+\frac{2}{x})}{x^3(2-\frac{1}{x^3})} = \lim_{x\to+\infty} \frac{(5+\frac{2}{x})}{x^2(2-\frac{1}{x^3})} = \frac{5}{+\infty} \to 0 = 0$$

$$\lim_{x\to+\infty} \frac{-5x+2}{2x^3-1} = \lim_{x\to\infty} \frac{x(-5+2/x)}{x^3(2-1/x^3)}$$

$$\lim_{x\to+\infty} \frac{(-5+2/x)}{x^2(2-1/x^3)} = \frac{-5}{[+\infty]} \to 0$$

$$\frac{-5}{10}, \quad \frac{-5}{100} \to, \quad \frac{-5}{10000} \to 0$$

$$\partial p < \partial q \Rightarrow \lim_{x\to+\infty} \frac{p(x)}{q(x)} = 0$$

(c) $\partial p > \partial q$

$$\lim_{x\to+\infty} \frac{2x^3-1}{5x+2} =$$

$$f(x) = \frac{x^3(2-\frac{1}{x^3})}{x[5+\frac{2}{x}]} = \frac{x^2(2-\frac{1}{x^3})}{(5+\frac{2}{x})}$$

x	$1/x^3$
1	1
10	$1/10^3 \to 0$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2(2-1/x^3)}{(5+\frac{2}{x})} = [+\infty]\cdot\frac{2}{5} = +\infty$$

$$\partial p > \partial q \Rightarrow \lim_{x\to+\infty} \frac{p(x)}{q(x)} = +\infty$$
$+ a_n > 0$

3.2 $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ (Rationella men inte bara med polynomial som faktorer)

En gång till behöver vi simplifiera: när man vill räkna $\lim_{x\to+\infty} \dfrac{x}{\sqrt{x^2+1}}$

räkna $\lim\limits_{\boxed{x \to +\infty}} \dfrac{x}{\underbrace{\sqrt{x^2+1}}}$

$$f(x) = \frac{x}{\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} = \frac{x}{\underbrace{\sqrt{x^2}}\left(\sqrt{1+\frac{1}{x^2}}\right)} \stackrel{*}{=}$$

$$f(x) = \frac{x}{\underbrace{|x|}\left(\sqrt{1+\frac{1}{x^2}}\right)}$$

Man vill räkna $\lim\limits_{\boxed{x \to +\infty}} f(x)$, då är $\underline{x>0}$ alltid.

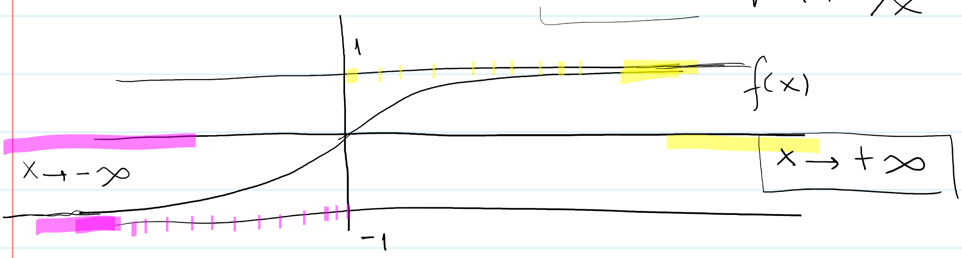
Då $\boxed{|x| = x}$, när $\underline{x>0}$ $\qquad \underbrace{x \to \infty}$

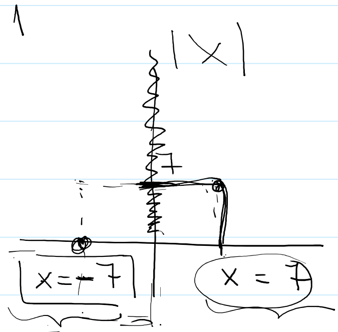
$$\lim_{\boxed{x \to +\infty}} f(x) = \lim_{x \to +\infty} \frac{\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}\right)} =$$

$$\lim_{x \to +\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} \quad \frac{1}{x^2} \to 0 \qquad = \frac{1}{1} = 1$$

Obs $\lim\limits_{\boxed{x \to -\infty}} f(x) = \lim\limits_{x \to -\infty} \dfrac{\cancel{x}}{(-\cancel{x})\left(\sqrt{1+\frac{1}{x^2}}\right)} =$ $\boxed{\begin{array}{c} -x > 0 \\ \text{när } x<0 \end{array}}$

$$= \lim_{\boxed{x \to -\infty}} \frac{-1}{\sqrt{1+1/x^2}} = -1$$





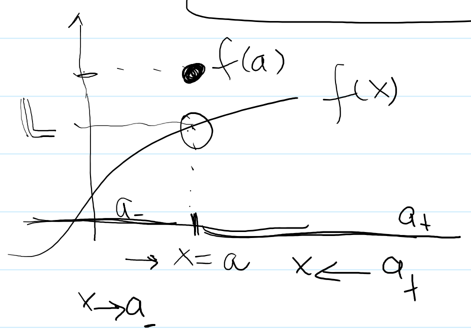
$x = \underbrace{-7} \to \boxed{-}(-7) \to +7$

Sidan 79 (Adams Bok) Kontinuitet

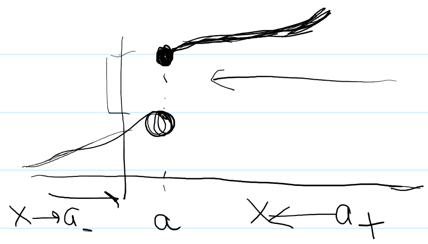
Def 4. f är kontinuerlig i en punkt a,

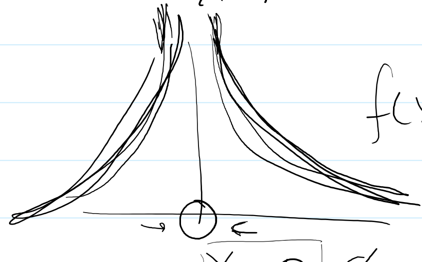
om $\lim f(x) \quad f(a)$

om $\boxed{\lim_{x\to a} f(x) = f(a)}$



När = $\boxed{L = f(a)}$ $= \lim_{x\to a} f(x)$ } f är kontinuerlig i a.





$f(x) = \frac{1}{x^2}$

$\boxed{x = 0} \notin D_f$

Sidan 80

Satsen 5

$\underbrace{\boxed{\lim_{x\to a} f(x) = f(a)}}_{\text{f är kontinuerlig i punkten a}} \iff \underbrace{\boxed{\lim_{x\to a_-} f(x) = \lim_{x\to a_+} f(x) = f(a)}}_{\text{f är höger och vänster kontinuerlig i punkten a}}$

f är kontinuerlig i punkten a

f är höger och vänster kontinuerlig i punkten a

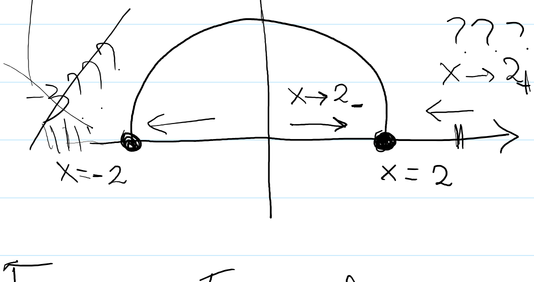
Sidan 80

Exempel $f(x) = \sqrt{4 - x^2}$ $\qquad \boxed{D_f = [-2, 2]}$

$f(-3) = \sqrt{4-(-3)^2}$

$\sqrt{4-9}$

$\sqrt{-1}$???



$\lim_{x\to 2_-} f(x) = f(2) = \sqrt{4-(2)^2}$

$= \sqrt{4-4} = 0$

Ex: Exempel av kontinuerlig funktioner $\qquad \sqrt{4-(-2)^2}$

$\sqrt{4-4}$

$p(x) = \text{polynomial} = a_n x^n + \dots + a_1 x + a_0 \qquad a_n \neq 0 \qquad \sqrt{0}$

$n = 2p$

$\lim p(x) = p(a) \qquad$ F. 1.1

$$\boxed{\lim_{x \to a} p(x) = p(a)}$$ F. 1.1

$p(x)$ är Kontinuerlig i en punkt a

på en Intervall

på hela Definitionsmängd

(b) rationella funktioner

(c) trigonometriska f. $\begin{cases} \cos(x) \\ \sin(x) \\ tg(x) \end{cases}$ ← Bara titta på D_f

Sidan 82

Satsen 6

$f: D_f \to V_f$

$g: D_g \to V_g$

} väl definerad i deras D_f och D_g

$c \in D_f$, $c \in D_g$

Om f och g är Kontinuerlig i punkten c

Då

1) $f(x) \pm g(x)$ kontinuerlig i c.

2) $\alpha f(x)$ kontinuerlig i c

3) $\dfrac{f(x)}{g(x)}$ kontinuerlig i c, $g(x) \neq 0$

$f(g(x))$ f är kontinuerlig i c

g är kontinuerlig i c

$x \longrightarrow g(x) \xrightarrow{f} f(g(x))$

$x \to c \longrightarrow g(c) \longrightarrow \boxed{\lim_{x \to c} f(g(x))} = f\left(\lim_{x \to c} g(x)\right)$

$= f(g(c))$

$f(x) = \sqrt{x}$

$f(x)=\sqrt{x}$

$\sqrt{\underbrace{(x^3+x)}_{g(x)}}$

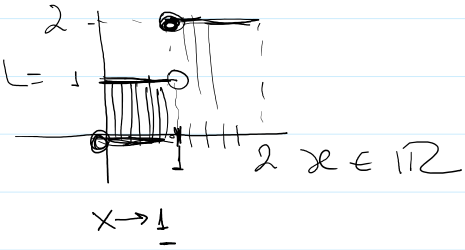
$$= f(g(c))$$

$$f(g(x)) = \sqrt{g(x)}$$

$$= \sqrt{(x^3+x)}$$

$x=1$

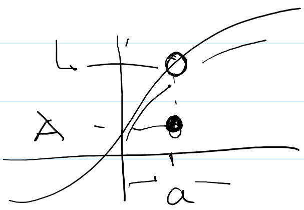
$$\lim_{x\to 1} f(g(x)) = f\left(\lim_{x\to 1} g(x)\right) = \sqrt{(1)^3+(1)} = \sqrt{2}$$



$$\lim_{x\to 1_-} f(x) = f(1_-) = 1$$

$$\neq$$

$$\lim_{x\to 1_+} f(x) = f(1_+) = 2$$

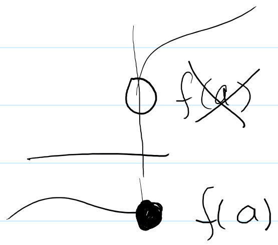


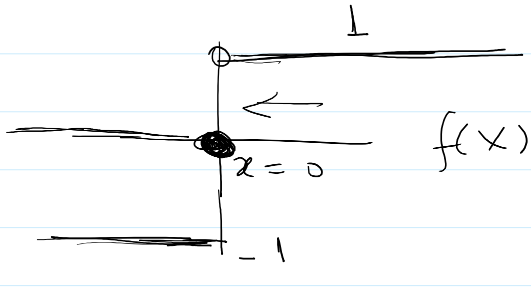
$$f(a) = A \neq L$$

$$\lim_{x\to a} f(x) = L$$

$x \to a_+$

$x \to a_-$





$f(x)=0$