Föreläsning 2.4

den 11 september 2020 07:52

$\boxed{f(x) = x^n}$ Compute by definition $f'(x)$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

Obs: $$\boxed{(a^n - b^n) = (a-b)\underbrace{(a^{n-1} + a^{n-2}\cdot b^1 + a^{n-3}\cdot b^2 + \dots + ab^{n-2} + b^{n-1})}_{n \text{ terms}}} \circledast$$

So assuming $a = x+h$, $b = x$, we have:

$$(x+h)^n - x^n) = \overbrace{(h)}^{a-b}\left((x+h)^{n-1} + (x+h)^{n-2}\cdot x^1 + (x+h)^{n-3}\cdot x^2 + \dots + (x+h)x^{n-2} + x^{n-1}\right) \circledast$$

$$\lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right) = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}}\left((x+h)^{n-1} + (x+h)^{n-2}\cdot x^1 + \dots + (x+h)x^{n-2} + x^{n-1}\right) =$$

$$\downarrow$$

$$x^{n-1} + x^{n-2}\cdot x^1 + \dots + (x)x^{n-2} + x^{n-1}$$

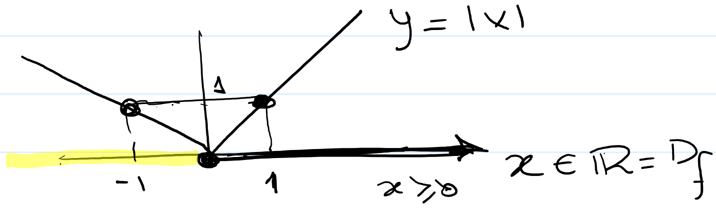
$$\parallel$$

$$\underbrace{x^{n-1} + x^{n-1} + \dots + x^{n-1} + x^{n-1}}_{n \text{ terms}} =$$

$$= n\,(x)^{n-1}$$

Conclusion $\boxed{f(x) = x^n} \longrightarrow \boxed{f'(x) = n\,x^{n-1}}$

Exemple (from F. 2.3)



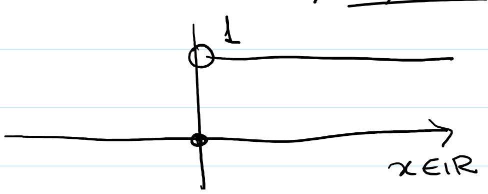
$f(x) = |x| \qquad x \in D_f$

$x = x^1$

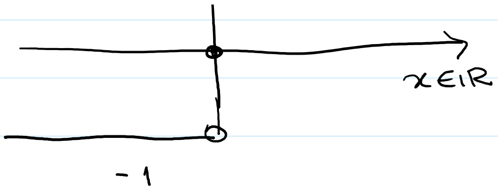
$$= \begin{cases} x, & \boxed{x \geq 0} \\ -x, & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} (1)\,x^{1-1} = (1)\cdot x^0 = 1, & \forall\, x > 0 \\ 1\,(-x)^{1-1} = -(1)\,x^0 = -1, & \forall\, x < 0 \end{cases}$$

? $f'(0)$? ?? $\qquad \nexists\, f'(0)$



$f'(x) = \{ +1, \; x > 0$



$$f'(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$\nexists f'(0)$ = it does not exist $\underline{f'(0)}$.

Differentials

Newton Quotient $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)} \quad \left(\dfrac{f(x_1) - f(x_2)}{x_1 - x_2} \right)$

$$\boxed{\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right)} = \lim_{\underbrace{\Delta x \to 0}_{h \to 0}} \left(\frac{f(x+h) - f(X)}{h} \right)$$

$$\boxed{f'(x)}$$

$$\frac{dy}{dx}$$

$$? \ \boxed{\frac{dy}{dx}} = f'(x) = \lim_{\Delta x} \left(\frac{\Delta y}{\Delta x} \right) \ ?$$

? Is this also a quociont ? $\longleftarrow$ Quotient ✓

$$\boxed{\frac{dy}{dx} = f'(x)} \implies \boxed{dy = f'(x) . dx}$$

<u>Differential</u>

Adams

Mål: <u>Differentiation Rules</u> $\quad$ § 2. 3 $\quad$ § 2. 4

<u>Theo1</u> $\quad$ f differentiable $\implies$ f is continuous

$f'(x) \implies \lim_{x \to a} f(x) = f(a)$

$\not\Leftarrow$ exemple $f(x) = |x|$ continuous but $\nexists f'(0)$

<u>Theo2</u> $\quad$ f, g differentiable $\implies$

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(f-g)'(x) = f'(x) - g'(x)$$

Theo 3 $\quad C$ = constant value $(C \in \mathbb{R})$

f differentiable $(\exists f')$

$$(Cf)'(x) = C(f'(x))$$

Theo 4 $\quad f, g$ differentiable

$$\boxed{(fg)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)} \quad *$$

$$f'(x) \cdot g(x) = (f \cdot g)'(x) - f(x) \cdot g'(x) \leftarrow$$

Theo 5

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}$$

Theo 6

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) =$$

$$f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \left(\frac{1}{g(x)}\right)' =$$

$$\frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) =$$

$$\boxed{\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}} = \left(\left(\frac{f}{g}\right)(x)\right)'$$

Proofs :

Theo 1. $\boxed{f \text{ differentiable at } x} \Rightarrow \boxed{f \text{ continuous at } x}$

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Proof:

By Hypothesis

f diffirentiable at x $\Rightarrow$ $\underbrace{\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = f'(x)}$ exists

I can use this result.

? f is continuous at x ?

$$\left(\boxed{\lim_{x\to a} f(x) = f(a)}\right) \text{ want}$$

$$\lim_{h\to 0} (f(x+h) - f(x))$$

$$\left(\lim_{x\to a} f(x) - f(a) = 0\right) \text{ want}$$

$$\left(\boxed{\lim_{h\to 0} f(x+h) - f(x)}\right) \text{ want}$$

$$\boxed{\lim_{h\to 0}(f(x+h)-f(x))1} = \lim_{h\to 0}(f(x+h)-f(x))\frac{h}{h} =$$

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\cdot h \overset{*}{=} \underbrace{\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)}_{f'(x)}\cdot\underbrace{\lim_{h\to 0} h}_{0}$$

$$\underbrace{f'(x)\cdot 0}_{\substack{\| \\ 0}}$$

$$\boxed{\lim_{h\to 0}(f(x+h)-f(x)) = \dots = 0} \Longrightarrow$$

$$\boxed{\lim_{h\to 0} f(x+h) = f(x)}$$ f is continuous at x □

Theo 4 f, g diffirentiable at x $\Longrightarrow$ $\boxed{(f\cdot g)'(x) = f'(x)g(x) + f(x)\cdot g'(x)}$

Proof: $$(f\cdot g)'(x) = \lim_{h\to 0}\left[\frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}\right] =$$

$$\lim_{h\to 0}\left[\frac{f(x+h)\cdot g(x+h) + 0 - f(x)\cdot g(x)}{h}\right]$$

$$\lim_{h\to 0}\left[\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}\right]$$

$$\lim_{h\to 0}\left[\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}\right]$$

$$\lim_{h\to 0}\left[\frac{(f(x+h)-f(x))\,g(x+h) + f(x)\,(g(x+h)-g(x))}{h}\right]$$

$$\lim_{h\to 0}\left[\underbrace{\left(\frac{f(x+h)-f(x)}{h}\right)}\cdot g(x+h)\right] + \lim_{h\to 0}\left(\underbrace{f(x)}\cdot\underbrace{\left(\frac{g(x+h)-g(x)}{h}\right)}\right) =$$

$$(f\cdot g)'(x) = \boxed{f'(x)\cdot g(x) + f(x)\cdot g'(x)} \quad \square$$

Theo 5 f diff $\Rightarrow \left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f^2(x)}$

Proof: We use the definition again.

$$\left(\frac{1}{f(x)}\right)' = \frac{d}{dx}\left(\frac{1}{f}\right)(x) = \lim_{h\to 0}\left(\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}\right)$$

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)-f(x+h)}{f(x+h)f(x)}\right) = \lim_{h\to 0} -\underbrace{\left(\frac{f(x+h)-f(x)}{h}\right)}\frac{1}{f(x+h)f(x)}$$

$$-\lim_{h\to 0}\underbrace{\left(\frac{f(x+h)-f(x)}{h}\right)}\cdot\frac{1}{f(x+h)f(x)} = \frac{-f'(x)}{f(x)\cdot f(x)} =$$

$$\Rightarrow \left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f^2(x)} \quad \square$$

The Chain Rule

Theor. 7

$\boxed{\text{IF}}$
$\boxed{\begin{array}{l} f(u) \text{ differentiable at } u = g(x) \\ g(x) \text{ differentiable at } x \end{array}}$

$\boxed{\text{Then}}$
The composite function $(f \circ g)(x) = f(g(x))$
is differentiable at x

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Explanation by the book.

In terms of Leibniz Notation.

$y = f(u)$ & $u = g(x) \Rightarrow y = f(g(x))$

$\frac{dy}{du}$ at $u \Rightarrow y$ changes $\frac{dy}{du}$

at x u changes $\frac{du}{dx}$

Now at x y changes $\frac{dy}{du} \times \frac{du}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

differential approach to the Chain Rule.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x} \qquad \Delta x \to 0$$

Ex: $y = \sqrt{u}$ $\qquad u = g(x)$

$g(x) = (x^2 + 1)$

$y = \sqrt{u} = u^{1/2}$

$(y \circ g) = \sqrt{(x^2+1)}$

$y' = \frac{1}{2} u^{1/2 - 1} =$

$$\boxed{(y \circ g) = \sqrt{(x^2+1)}}$$

$$y' = \frac{1}{2} u^{\frac{1}{2}-1} =$$

$$y' = \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{u}}$$

$$(y \circ g)'(x) = ? \quad \underbrace{\frac{dy}{du}}_{\frac{1}{2}\frac{1}{\sqrt{u}}} \circ \frac{du}{dx} =$$

$$\frac{1}{2}\frac{1}{\sqrt{u}} \cdot \frac{dg(x)}{dx}$$

$$g(x) = x^2 + 1$$

$$g'(x) = 2x + 0$$

$$g'(x) = 2x$$

$$\Downarrow$$

$$(y \circ g)'(x) = \frac{1}{\cancel{2}}\frac{1}{\sqrt{x^2+1}} \cdot (\cancel{2}x) = \frac{x}{\sqrt{x^2+1}}$$

$$H(x) = \underbrace{\sqrt{x^3 + 1/x}} + \underbrace{x}$$

$$(x^1)'$$

$$H'(x) = \underbrace{\left(\sqrt{x^3+\frac{1}{x}}\right)}_{A = (f \circ g)'(x)} + \frac{dx}{dx}$$

$$= 1$$

$$1\, x^{1-1}$$

$$1\, \cancel{x}^0 = 1$$

$$A \quad \sqrt{g(x)} \quad ; \quad x^3 + \frac{1}{x} = g(x)$$

$$\boxed{(f \circ g)'(x) = f(g(x)) = f'(g(x)) \cdot g'(x)}$$

$$f(u) = \sqrt{u} \qquad f'(u) = (u^{\frac{1}{2}})'$$

$$= \frac{1}{2} u^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{u}}$$