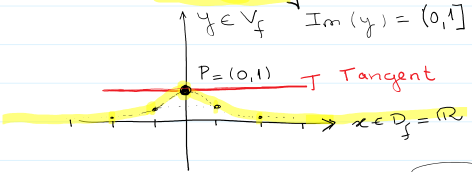
Övningar A 2.1 – 6 / 21 sidan 100
A 2.2 – 21 / 48 sidan 107 – 108
A 2.3 – 18 sidan 115

6. Find Equation of the Straight line tangent to the given curve at the point indicated $f(x) = \boxed{y = \dfrac{1}{x^2+1}}$ $P = (0,1)$

a)

x	$y(x)$
-2	$1/5$
-1	$1/2$
0	1
1	$1/2$
2	$1/5$

$y \in V_f$ $\quad Im(y) = [0,1] \subseteq V_f$



Tangent: $\underset{y(x)}{t(x)} = m(x - \underset{x_0}{0}) + \underset{y_0}{1}$ $\quad P = (x_0, y_0) = (0,1)$

$m = y'(0)$ $\quad m = \lim_{h\to 0}\left[\dfrac{f(x+h) - f(x)}{h}\right] = \lim_{h\to 0} \dfrac{\left(\dfrac{1}{(0+h)^2+1} - 1\right)}{h}$ $\quad f(x)$, $x=0$

$m = \lim_{h\to 0}\left[\dfrac{1}{h^2+1} - 1\right]\cdot\dfrac{1}{h} = \lim_{h\to 0}\left(\dfrac{1-(h^2+1)}{h^2+1}\right)\dfrac{1}{h}$

$m = \lim_{h\to 0}\left(\dfrac{-h^{\cancel{2}}}{h^2+1}\right)\cdot\dfrac{1}{\cancel{h}} = \lim_{h\to 0}\left(\dfrac{-h}{h^2+1}\right) = \dfrac{-0}{1} = 0$

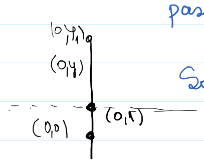
$m = 0 \Rightarrow t(x) = 0(\cancel{x-0}) + 1$ $\quad \boxed{t(x) = 1 \quad (y=1)}$

All points on this tangent are given by $\boxed{P = (x, 1)}$ or $\boxed{t(x) = 1}$

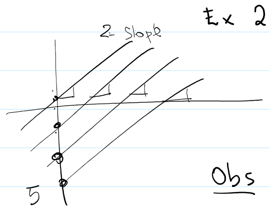
If we want to compute the normal line: $\boxed{y^N(x) = \left(-\dfrac{1}{m}\right)(x - 0) + 1}$,

But $m = 0 \Rightarrow -\dfrac{1}{m} = -\dfrac{1}{0}$???

So the Normal line is parallel to the y-axis passing at the Point $P = (0,1)$



So y^N: $\boxed{x = 0}$. (All point on this line has coordinate $Q = (0, y)$ $y \in \mathbb{R}$ □



Ex 2.1 Find all points on the curve $\boxed{y = x^3 - x + 1}$ where the tangent line is parallel to the line $y = 2x + 5$

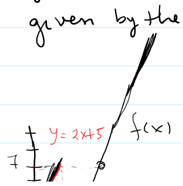
Obs: $y(x) = x^3 - x + 1$ is our $f(x)$
$y(x) = 2x + 5$ is another function, just parallel to the Tangent we want to compute.

Please! Don't mixture those two $y(x)$. They are different things with the same name $y(x)$ given by the book.

1°) Some intuition about $\boxed{f(x) = x^3 - x + 1}$

$f(x) = x^3 - x + 1$

x	$f(x) = x^3 - x + 1$
-3	$(-3)^3 + 3 + 1 < 0$



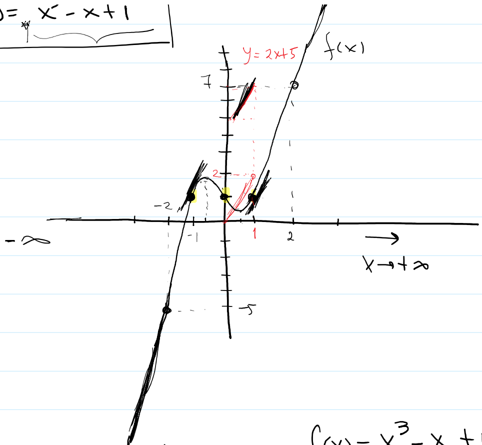
1°) Some intuition about $f(x) = x^3 - x + 1$

$f(x) = x^3 - x + 1$

$f'(x) = 3x^2 - 1 + 0$

$f'(x) = 3x^2 - 1$

x	$f(x) = x^3 - x + 1$
-3	$(-3)^3 + 3 + 1 < 0$
-2	$(-2)^3 + 2 + 1 < 0$
-1.5	$(-1.5)^3 + 1.5 + 1 < 0$
-1	$(-1)^3 + 1 + 1 = 1$
-0.5	$(-\frac{1}{8}) + \frac{1}{2} + 1 = 1 + 3/8$
0	1
0.5	$(\frac{1}{8}) - \frac{1}{2} + 1 = 1 - 3/8$
1	1
1.5	$(1.5)^3 - 1.5 + 1 > 0$
2.0	$8 - 2 + 1 > 0$
3.0	$(3)^3 - 3 + 1 > 0$



$f(x) = x^3 - x + 1$

Let's compute the slope of the curve $y = f(x)$

$f(x) = x^3 - x + 1$

$f(x+h) = (x+h)^3 - (x+h) + 1$

$$\boxed{m = f'(x)} = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h) + 1] - [x^3 - x + 1]}{h}$$

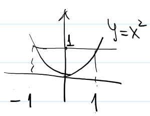
$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{-x} - h \cancel{+1} \cancel{-x^3} \cancel{+x} \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}} =$$

$$= \lim_{h \to 0} 3x^2 - 1 + 3x\cancel{h}^{0} + \cancel{h^2}^{0} = \boxed{3x^2 - 1}$$

So $m = 3x^2 - 1$ is the curve slope.

But we want to find points on the curve $y = f(x)$ where the tangent line is parallel to $y = 2x + 5$



So $3x^2 - 1 = 2 \Rightarrow 3x^2 = 2 + 1 = 3$

$\cancel{3}x^2 = \cancel{3} \Rightarrow \boxed{x^2 = 1} \Rightarrow x = \pm 1$

So there are 2 points:

$P_1 = (1, f(1))$ $\quad P_2 = (-1, f(-1))$

$P_1 = (1, \cancel{1^3} - \cancel{1} + 1)$ $\quad P_2 = (-1, \cancel{(-1)^3} + \cancel{1} + 1)$

$\boxed{P_1 = (1, 1)}$ $\quad \boxed{P_2 = (-1, 1)}$

T_1: $t(x) = 2(x-1) + 1$ $\quad t(x) = 2(x+1) + 1$

$y(x)$

□

A 2.2 Ex 21 page 107

21:(a) Calculate the derivative of $F(x)$ from the DEFINITION

(b) Express the result from (a) using differentials. (Notation)

(a) $F(x) = \dfrac{1}{\sqrt{1 + x^2}}$

by definition $F'(x) = \lim_{h \to 0} \left[\dfrac{F(x+h) - F(x)}{h} \right]$

Let's compute $\frac{1}{h}[F(x+h) - F(x)]$, then we compute the limit.

$\frac{1}{h}[F(x+h) - F(x)] = \frac{1}{h}\Big[\ \frac{1}{\ }\ \ \frac{1}{\ }\ \Big]$

Let's compute $\frac{1}{h}[F(x+h)-F(x)]$, then we compute the limit.

$$\frac{1}{h}\left[F(x+h)-F(x)\right] = \frac{1}{h}\left[\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{h}\left[\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}\right] \cdot \frac{\left[\frac{1}{\sqrt{1+(x+h)^2}} + \frac{1}{\sqrt{1+x^2}}\right]}{\left[\frac{1}{\sqrt{1+(x+h)^2}} + \frac{1}{\sqrt{1+x^2}}\right]}$$

$$\frac{1}{h}\left[F(x+h)-F(x)\right] = \frac{1}{h}\left[\underbrace{\frac{1}{(1+(x+h)^2)} + \cancel{\frac{1}{(\ )(\ )}} - \cancel{\frac{1}{(\ )(\ )}} - \frac{1}{(1+x^2)}}_{\text{same denominator}}\right] \cdot \frac{1}{\left[\frac{1}{\sqrt{1+(x+h)^2}} + \frac{1}{\sqrt{1+x^2}}\right]}$$

same denominator $\Downarrow$

$$\frac{\left[\frac{(1+x^2) - (1+(x+h)^2)}{(1+(x+h)^2)(1+x^2)}\right]}{h\left[\frac{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}{\sqrt{1+(x+h)^2}\cdot\sqrt{1+x^2}}\right]} =$$

$$\frac{1}{\frac{a}{b}} = 1 \cdot \frac{b}{a}$$

$$= \frac{1}{h}\left[\frac{(1+x^2) - (1+(x+h)^2)}{(1+(x+h)^2)^{\cancel{1}}(1+x^2)^{\cancel{1}}}\right] \cdot \left[\frac{\cancel{(1+(x+h)^2)^{1/2}} \cdot \cancel{(1+x^2)^{1/2}}}{\sqrt{1+x^2} + \sqrt{1+(x+h)^2}}\right]$$

$$\frac{(\ )^{1/2}}{(\ )^{1}} \qquad \frac{1}{(\ )^{1-1/2}}$$

$$= \frac{1}{h}\left[\frac{\cancel{1+x^2} - (\cancel{1}+\cancel{x^2}+2hx+h^2)}{(1+(x+h)^2)^{1/2}(1+x^2)^{1/2}}\right]\left[\frac{1}{\sqrt{1+x^2}+\sqrt{1+(x+h)^2}}\right]$$

$$= \frac{1}{\cancel{h}}\left[\frac{-2x\cancel{h} - h^{\cancel{2}}}{(1+(x+h)^2)^{1/2}(1+x^2)^{1/2}}\right] \cdot \left[\frac{1}{\sqrt{1+x^2}+\sqrt{1+(x+h)^2}}\right]$$

$$\text{So } \lim_{h\to 0} \frac{-2x - h^{\to 0}}{\left[(1+(x+h)^2)^{1/2}\cdot(1+x^2)^{1/2}\right]\left[\sqrt{1+x^2}+\sqrt{1+(x+h)^2}\right]} =$$

$$= \frac{-2x}{\left[(1+x^2)^{1/2}\cdot(1+x^2)^{1/2}\right]\cdot\left[\underbrace{\sqrt{1+x^2}+\sqrt{1+x^2}}\right]} =$$

$$\sqrt{a} = a^{1/2}$$

$$= \frac{-\cancel{2}x}{(1+x^2)^1 \cdot \cancel{2}\sqrt{1+x^2}} =$$

$$= \frac{-x}{(1+x^2)\cdot(1+x^2)^{1/2}} = \frac{-x}{(1+x^2)^{1+1/2}} =$$

$$\boxed{F'(x) = \frac{-x}{(1+x^2)^{3/2}}}$$

A will compute the same $F'(x)$, but using now the Derivation rules:

$$F(x) = \frac{1}{\sqrt{1+x^2}} = \boxed{(1+x^2)^{-1/2}} \qquad u^{-1/2}$$

$$u = (1+x^2)$$

$F'(x)$ must be computed by the Chain rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x)$$

$$-\tfrac{1}{2}-1$$

$F'(x)$ must be computed by the Chain rule $\quad u = (1+x^2)$

$(f(g(x)))' = f'(g(x)) \cdot g'(x) \quad\quad -\frac{1}{2} u^{-\frac{1}{2}-1}$

So

$$F'(x) = \left(-\frac{1}{2}\right) \underbrace{(1+x^2)^{-\frac{1}{2}-1}}_{f'(g(x))} \cdot \underbrace{(2x)}_{g'(x)}$$

$$\boxed{F'(x) = -\frac{1}{\cancel{2}} \cancel{2}x \frac{1}{(1+x^2)^{+3/2}} = \frac{-x}{(1+x^2)^{3/2}}}$$

I prefer this method!

b) $\quad \dfrac{dF}{dx} = F'(x) \iff dF = F'(x) \cdot dx$ $\quad$ Differential

$$\boxed{dF = \left(\frac{-x}{(1+x^2)^{3/2}}\right) \cdot dx}$$

Ex 48 page 108 $\quad$ A 2.2

Find eq of 2 straight lines that have slope (-2) and are tangent to the graph $y = \frac{1}{x}$

Resolution:

Since we don't know the points, we begin computing the slope of the curve:

$$y'(x) = \lim_{h\to 0} \frac{\left[\frac{1}{x+h} - \frac{1}{x}\right]}{h} = \lim_{h \to 0} \frac{\left[\frac{x-(x+h)}{x(x+h)}\right]}{h} =$$

$$\lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{-\cancel{h}}{x(x+\cancel{h})}\right] = \frac{-1}{x \cdot x} = \frac{-1}{x^2}$$

The slope must be -2 (according to the exercise)

so $\quad y'(x) = \cancel{-}\frac{1}{x^2} = \cancel{-}2 \Rightarrow \frac{1}{x^2} = 2 \Rightarrow x^2 = \frac{1}{2}$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$P_1 = \left(\frac{1}{\sqrt{2}}, f\left(\frac{1}{\sqrt{2}}\right)\right) \quad P_2 = \left(-\frac{1}{\sqrt{2}}, f\left(-\frac{1}{\sqrt{2}}\right)\right) \quad f(x) = \frac{1}{x}$

$P_1 = \left(\frac{1}{\sqrt{2}}, \sqrt{2}\right) \quad P_2 = \left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$

$T_1: \; y(x) = m(x-x_0) + y_0 \quad\quad T_2: \; y(x) = m(x-x_1) + y_1$

$y(x) = -2\left(x - \frac{1}{\sqrt{2}}\right) + \sqrt{2} \quad\quad y(x) = -2\left(x + \frac{1}{\sqrt{2}}\right) - \sqrt{2}$ $\quad \square$

Obs: just to compute $f'(x)$ $\quad f(x) = \frac{1}{x}$ using the rules:

$f(x) = \frac{1}{x} = x^{-1}$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f'(x) = -1\,(x)^{-1-1} \cdot 1 = -\frac{1}{x^2}$$

done!

Å.2.3 Ex 18

Compute $g'(u)$ simplifying as much as possible.

$$\boxed{g(u) = \frac{u \cdot \sqrt{u} - 3}{u^2}}$$

Before we derivate : $g(u) = \dfrac{u \cdot u^{1/2} - 3}{u^2} =$

$$g(u) = \left(u^{3/2} - 3\right)\left(u^{-2}\right) =$$

$$g(u) = u^{3/2-2} - 3u^{-2} =$$

$$\boxed{g(u) = \underbrace{u^{-1/2}}\,-\,\underbrace{3u^{-2}}}$$

Now $g'(u) = \underbrace{-\frac{1}{2}\,u^{-1/2-1}} - \underbrace{3\,(-2)\,u^{-2-1}}$

$$\boxed{g'(u) = -\frac{1}{2}\,u^{-3/2} + 6u^{-3}}$$

Just for fun, try to compute $g'(u)$ applying the rules directly to $g(u) = \dfrac{u\sqrt{u} - 3}{u^2}$ and compare ...

Thank you for today!

$$H(x) = \sqrt{x^3 + \frac{1}{x}} + x$$

$$H'(x) = \underbrace{\left(\sqrt{x^3 + 1/x}\right)'}_{\text{Chain Rule}} + \underbrace{(x)'}_{=1}$$

$f(g(x))$

$f(u)$, $u = g(x)$

$$f(g(x)) = \sqrt{x^3 + \frac{1}{x}} \qquad (f(g(x)))' = \underbrace{f'(g(x))}_{\frac{df(u)}{du}} \cdot g'(x)$$

$$f(u) = \sqrt{u} = u^{1/2}$$

$$u = g(x) = \left(x^3 + \frac{1}{x}\right)$$

$$\frac{df}{du} = \left(u^{1/2}\right)' = \frac{1}{2}\,u^{1/2-1} = \frac{1}{2}\,u^{-1/2} = \frac{1}{2\sqrt{u}}$$

$$g'(x) = \left(x^3 + \frac{1}{x}\right)'$$

$$(x^3)' + (x^{-1})'$$

$$3x^2 - 1\,x^{-1-1}$$

$$\underbrace{3x^2 - 1\,x^{-2}}$$

$$(f \circ g)'(x) = \frac{1}{2\sqrt{x^3 + \frac{1}{x}}} \cdot \left(3x^2 - \frac{1}{x^2}\right)$$

$$(f\circ g)'(x) = \frac{1}{2\sqrt{x^3+\frac{1}{x}}} \cdot \left(3x^2 - \frac{1}{x^2}\right) \qquad 3x^2 - 1x^{-2}$$

$X = 0 \notin D_{f'}$ $\qquad x^3 + \frac{1}{x} \neq 0$ because it is on the denominator of the expression.

$F(x) = \left(2\sqrt{x} + \frac{3}{x}\right) \cdot x^7$ $\qquad$ Obtain $F'(x)$ using the rules.

$F(x) = f(x) \cdot g(x)$ $\qquad \begin{cases} f(x) = 2\sqrt{x} + \frac{3}{x} \; \circledast \\ g(x) = x^7 \end{cases}$

$$F'(x) = (f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$\circledast$ $\boxed{f(x) = 2(x)^{1/2} + 3 \cdot x^{-1}}$

$f'(x) = \cancel{2} \cdot \frac{1}{\cancel{2}}(x)^{\frac{1}{2}-1} + 3 \cdot (-1)\, x^{-1-1}$

$f'(x) = x^{-1/2} - 3x^{-2}$ $\qquad \boxed{f'(x) = x^{-1/2} - 3x^{-2}}$

$g(x) = x^7$

$g'(x) = 7x^6$

$$F'(x) = (x^{-1/2} - 3x^{-2}) \cdot x^7 + (2x^{1/2} + 3x^{-1})(7 \cdot x^6)$$

$$(x)^{-\frac{1}{2}+7} - 3x^{-2+7} + 14x^{\frac{1}{2}+6} + 21x^{-1+6}$$

$$1(x)^{13/2} - 3x^5 + 14x^{13/2} + 21x^5$$

$$(15)\, x^{13/2} + \underbrace{(21-3)}_{18} x^5$$

$$\boxed{F'(x) = 15x^{13/2} + 18x^5}$$