[Use ($\varepsilon-\delta$) definition to prove $\lim_{x\to 1}\sqrt{x} = 1$

ε δ definition

[$\lim_{x\to a} f(x) = L \iff \forall\, \varepsilon>0,\ \exists\, \delta=\delta(\varepsilon)>0 \,/\, 0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon.$]

Resolution

Given $\varepsilon > 0$

1) (We need to find $\delta>0$ depending on ε ($\delta=\delta(\varepsilon)$) such that $0<|x-a|<\delta$ implies that $|f(x)-L|<\varepsilon$.

2) (we start searching for δ by analysing exactly what we would like to conclude: $|f(x)-L|<\underline{\underline{\varepsilon}}$)

$$|f(x)-L| < \varepsilon$$

$$|\sqrt{x}-1| < \varepsilon \Rightarrow$$

$$-\varepsilon < \sqrt{x}-1 < \varepsilon \Rightarrow$$

$$1-\varepsilon < \sqrt{x} < 1+\varepsilon \Rightarrow$$

$$(1-\varepsilon)^2 < x < (1+\varepsilon)^2 \Rightarrow$$

$$1-2\varepsilon+\varepsilon^2 < x < 1+2\varepsilon+\varepsilon^2 \Rightarrow$$

$$1-(2\varepsilon-\varepsilon^2) < x < 1+(2\varepsilon+\varepsilon^2) \Rightarrow$$

$$\boxed{-(2\varepsilon-\varepsilon^2) < x-1 < (2\varepsilon+\varepsilon^2)}$$

Obs $0<\varepsilon<1 \Rightarrow 0<\varepsilon^2<\varepsilon$

So, If we assume $\delta=\delta(\varepsilon) = \min\{2\varepsilon-\varepsilon^2,\ 2\varepsilon+\varepsilon^2\} =$
$= 2\varepsilon-\varepsilon^2$

Then this is the δ-value that will garantee that all values satisfying $|x-1|<\delta$ will have $|f(x)-1|<\varepsilon$ $\quad f(x)=\sqrt{x}$

Let's confirm this

(and to confirm, I will start by $|x-a|<\delta$, and I will be able to conclude that $|f(x)-L|<\varepsilon$.)

$|x-a|<\delta$

$$|x-1|<\delta=(2\varepsilon-\varepsilon^2) \Rightarrow$$

$$-(2\varepsilon-\varepsilon^2) < x-1 < (2\varepsilon-\varepsilon^2) \Rightarrow$$

$$\underbrace{1-(2\varepsilon-\varepsilon^2)} < x < 1+2\varepsilon-\varepsilon^2 \Rightarrow$$

$$1-2\varepsilon+\varepsilon^2 < x < 1+2\varepsilon-\varepsilon^2 < \underbrace{1+2\varepsilon+\varepsilon^2}$$

$$(1-\varepsilon)^2 < x < (1+\varepsilon)^2 \Rightarrow$$

$$(1-\varepsilon) < \sqrt{x} < (1+\varepsilon) \Rightarrow$$

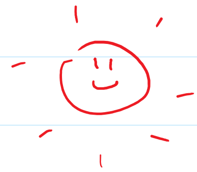
$$\boxed{-\varepsilon < \sqrt{x}-1 < \varepsilon} \Rightarrow$$

$$|\sqrt{x}-1| < \varepsilon$$

$$|f(x)-L| < \varepsilon \quad \square$$

So, usually we don't need to make the confirmation part. But now, that we are learning the tecknique, I do recommend to verify, testing the chosen δ and showing that it really implies what we need.

Below I put some links with more examples that might help! Lycka till!



https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(Apex)/01%3A_Limits/1.02%3A_Epsilon-Delta_Definition_of_a_Limit

https://socratic.org/questions/how-do-you-prove-that-the-limit-of-sqrt-x-7-3-as-x-approaches-2-using-the-epsilo

https://socratic.org/questions/how-do-you-prove-that-the-limit-of-sqrt-x-1-2-as-x-approaches-3-using-the-epsilo