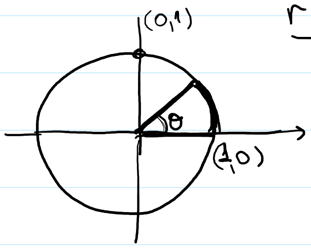
Mål: A §2.5 Derivator av Trigonometriska funktioner
Viktigt trigonometriskt gränsvärde
§2.6 Högre derivator
§2.7 Differentialer
Quiz – Comments & Answer.

Obs • P7 (pages 46 – 58) Adams Bok.

• page 47



$\underline{r=1}$ (example)

$\theta = t$ = angle

a) Area of the sector $\quad A_{\triangle} = \frac{\theta}{\cancel{2\pi}}(\cancel{\pi} r^2) = r^2\frac{\theta}{2}$

b) length of the arc s

$$S = \frac{\theta}{\cancel{2\pi}}(\cancel{2\pi} r) = r\theta$$

r = radius

• Hypothesis: arguments of trigonometric functions are measured in $\underline{\text{Radians}}$

$$2\pi \text{ radians} = 360^\circ \text{ degrees.}$$

:= by definition
≡ equivalent

Theorem 7 pag 121.

The functions cos(θ) & sin(θ) are continuous $\forall \theta \in \mathbb{R}$,

$$\lim_{\theta \to 0} \sin\theta = 0 = (\sin(0)) \quad / \quad \lim_{\theta\to 0} \cos(\theta) = (\cos(0)) = 1$$

by continuity

(*) Theorem 8

$$\boxed{\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1}$$

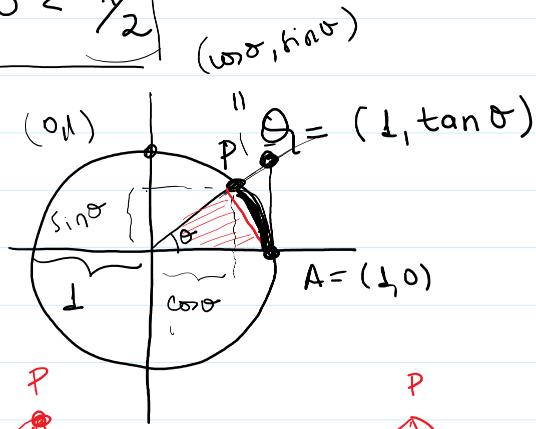
(this means that $\sin x \approx x$ when $x \to 0$) $\approx$ is approximated by

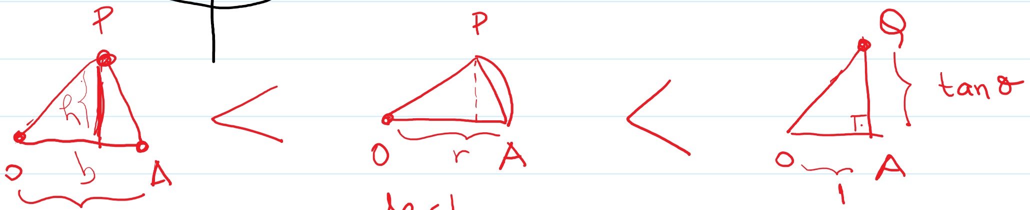
Proof: lim $\quad \left(\frac{\lim_{\theta\to 0}\sin\theta}{}\right) = \left[\frac{0}{}\right] ???$

Proof: lim $\qquad \left(\dfrac{\lim_{\theta \to 0} \sin\theta}{\lim_{\theta \to 0} \theta} \right) = \left[\dfrac{0}{0} \right]$???

So, we need to evaluate it by a different strategy.

Let $\boxed{0 < \theta < \pi/2}$





Sector

(pizza slice)

Areas
$$\boxed{\frac{1}{\cancel{2}}\,\overbrace{1}^{b}\cdot\overbrace{\sin\theta}^{h} < (1)^2 \cdot \frac{\theta}{\cancel{2}} < \frac{1}{\cancel{2}} \cdot 1 \cdot \tan\theta}$$

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta} \qquad (* \ \sin\theta \neq 0)$$

$$\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{\cos\theta}{\sin\theta}$$

$$1 > \overbrace{\left(\frac{\sin\theta}{\theta}\right)}^{*} > \cos\theta$$

$$\cos\theta < \left(\frac{\sin\theta}{\theta}\right) < 1$$

By the Squeeze Theorem, we have:

$$\underbrace{\lim_{\theta \to 0} \cos\theta}_{1} < \lim_{\theta \to 0} \frac{\sin\theta}{\theta} < \underbrace{\lim_{\theta \to 0} 1}_{1}$$

$$\underbrace{}_{\downarrow \; 1} \qquad \underbrace{}_{\downarrow \; 1}$$

By the Squeeze Theorem

$$\boxed{\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right) = 1}$$

$\square$

II $\displaystyle \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

By the Rules : $\dfrac{\lim_{\theta \to 0} (\cos\theta - 1)}{\lim_{\theta \to 0} \theta} = \dfrac{\cos(0) - 1}{0} = \dfrac{1-1}{0} = \left[\dfrac{0}{0}\right]$??

$\boxed{\cos^2\theta + \sin^2\theta = 1}$

Proof

$$\boxed{\frac{\cos\theta - 1}{\theta} * \frac{(\cos\theta + 1)}{(\cos\theta + 1)}} = \frac{\cos^2\theta + \cancel{\cos\theta} - \cancel{\cos\theta} - (1^2)}{\theta\ (\cos\theta + 1)} =$$

$$\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} =$$

$(a-b)(a+b) = a^2 - b^2$

$(\cos\theta - 1)(\cos\theta + 1) = \cos^2\theta - 1^2$

$\cos^2\theta = 1 - \sin^2\theta$

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = - \lim \underline{\qquad\qquad} = \cancel{1} - \sin^2\theta - \cancel{1}$$

$$- \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin\theta}{(\cos\theta + 1)}\right) = - \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right) \cdot \lim \qquad)$$

$$- 1 * \left(\frac{0}{1+1}\right) =$$

$$- 1 * 0 = 0$$

Theorem 9 $\boxed{\dfrac{d}{dx} (\sin x) = \cos x}$

Proof d ... = lim sin(x+h) - sin(x) ...

Proof: $$\frac{d}{dx}\sin x = \boxed{\lim_{h\to 0}\frac{\boxed{\sin(x+h)} - \sin(x)}{h}} = \left[\frac{\sin x - \sin x}{0}\right] = \left[\frac{0}{0}\right] ???$$

$$\sin(x+h) = \boxed{\sin x\cosh + \cos x \sinh}$$ (*) (P7 page 47-58)

$$\lim_{h\to 0}\left(\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}\right) =$$

$$\lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \cos x\frac{\sin h}{h} =$$

$$\sin x\underbrace{\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)}_{=0} + \cos x\underbrace{\lim_{h\to 0}\left(\frac{\sin h}{h}\right)}_{=1}$$

* Before Pause   * (Before Pause)

$$\cancel{\sin x\cdot 0} + \cos x\cdot 1 \Rightarrow$$

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$ □

Theo 10 $$\boxed{\frac{d}{dx}\cos x = -\sin x}$$ ←

By definition

Proof: $$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{(\cancel{x}+h) - \cancel{x}} =$$

$$= \lim_{h\to 0}\frac{\cos(x+\overset{\to 0}{h}) - \cos x}{h} = \left[\frac{\cos x - \cos x}{0}\right] = \left[\frac{0}{0}\right] ??$$

(*) $$\cos(x+h) \overset{\boxed{*}}{=} \cos(x)\cos h - \sin x\sin h$$

$$\lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h} =$$

$$\lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x\sin h}{h} =$$

a b

$$\lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} =$$

$$\lim_{h\to 0} \cos x \left(\frac{\cos h - 1}{h}\right) - \lim_{h\to 0} \frac{\overset{a}{\sin x}\,\overset{b}{\sin h}}{\underset{c}{h}} = \qquad \frac{ab}{c} = a\left(\frac{b}{c}\right)$$

$$\cos x \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) - \sin x \lim_{h\to 0}\left(\frac{\sin h}{h}\right) =$$

$$\cancel{\cos x \cdot 0} - \sin x \cdot 1 = -\sin x$$

Conclusion $\boxed{\frac{d}{dx}\cos x = -\sin x}$

Example 2 a) $\sin(\pi x) + \cos(3x)$ Compute $\frac{d}{dx}\cdot$

b) $x^2 \cdot \sin(\sqrt{x})$

c) $\frac{\cos x}{1 - \sin x}$

a) $\frac{d}{dx}\left(\overset{f(x)}{\sin(\pi x)} + \overset{g(x)}{\cos(3x)}\right) = \underbrace{\frac{d}{dx}\left(\sin(\pi x)\right)}_{\text{Chain Rule}} + \underbrace{\frac{d}{dx}\left(\cos(3x)\right)}_{\text{Chain Rule}}$

Chain Rule $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}\left(\sin(\pi x)\right) = \cos(\pi x) \cdot \pi = \pi \cdot \cos(\pi x)$$

$$\frac{d}{dx}\left(\cos(3x)\right) = -\sin(3x)(3) = -3\sin(3x)$$

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \pi\cos(\pi x) - 3\sin(3x) \qquad \square$$

b) $\frac{d}{dx}\left(\underset{f(x)}{x^2} \cdot \underbrace{\sin(\sqrt{x})}_{g(x)}\right) = f'(x)\cdot g(x) + f(x) \cdot \underbrace{g'(x)}_{\circledast}$

Chain Rule

$$\frac{d}{dx}(x^2) \cdot \sin(\sqrt{x}) + x^2 \cdot \frac{d}{dx}\left(\sin(\sqrt{x})\right) =$$

chain rule

$$\underbrace{\frac{d}{dx}(x^2)}\cdot\sin(\sqrt{x}) + x^2\cdot\frac{d}{dx}\left(\sin(\sqrt{x})\right) =$$

$\sqrt{x} = x^{1/2}$

$$2x^{2-1}\cdot\sin(\sqrt{x}) + x^2\cdot\cos(\sqrt{x})\cdot\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}$$

$$2x\cdot\sin(\sqrt{x}) + \frac{1}{2}x^2\cdot x^{-1/2}\cos(\sqrt{x}) =$$

$$2x\sin(\sqrt{x}) + \frac{1}{2}x^{3/2}\cdot\cos(\sqrt{x}) \quad \square$$

$(f\cdot g)' = f'\cdot g + f\cdot g'$

C) $\frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right) = \frac{d}{dx}\left(\cos x\cdot\frac{1}{(1-\sin x)}\right) =$

$$\underbrace{\frac{d}{dx}\cos x}\cdot\left(\frac{1}{1-\sin x}\right) + \cos x\cdot\frac{d}{dx}\left(\frac{1}{1-\sin x}\right)$$

$(u)^{-1}$

$-1\,u^{-1-1}\cdot u'$

$$-\sin x\cdot\left(\frac{1}{1-\sin x}\right) + \cos x\,\frac{d}{dx}\left(\underbrace{(1-\sin x)^{-1}}\right) =$$

$$\frac{-\sin x}{(1-\sin x)} + \cos x\left[(-1)\,(\underbrace{1-\sin x})^{-2}\cdot(-\cos x)\right] =$$

$(-1)\cdot(-1) = 1$

$$\frac{-\sin x}{(1-\sin x)} + \cos^2 x\,\frac{1}{(1-\sin x)^2} =$$

$$= \frac{-\sin x\,(1-\sin x) + \cos^2 x}{(1-\sin x)^2} = \frac{-\sin x + \overbrace{\sin^2 x + \cos^2 x}^{1}}{(1-\sin x)^2} =$$

$$= \frac{\cancel{(1-\sin x)}}{(1-\sin x)^{\cancel{2}}} = \frac{1}{(1-\sin x)}$$

* $\frac{d}{dx}(\tan x) = ?$

* $\frac{d}{dx}(\cot x) = ?$

$* \quad \frac{}{dx}(\ldots) = \ldots$

$* \quad \frac{d}{dx}\left(\cos x \sin x\right) = ?$

$* \quad \frac{d}{dx}\left(\sin(2x)\right) = ?$

High order Derivative

$$p(x) = x^3 + 2x^2 + 1$$

$$\boxed{\frac{d}{dx}p(x) = 3x^2 + 4x^1 + 0} = p'(x)$$

$$\frac{d}{dx}\left(p'(x)\right) = \frac{d}{dx}\left(3x^2 + 4x\right) = 3.2x + 4 = 6x + 4$$

$$p''(x) = 6x + 4$$

$$p^{(''')}(x) = \frac{d}{dx}\left(p''(x)\right) = \frac{d}{dx}\left(6x + 4\right) = 6.(1) + 0 = 6$$

$$\begin{cases} p(x) = x^3 + 2x^2 + 1 \\ p'(x) = 3x^2 + 4x \\ p''(x) = 6x + 4 \\ p'''(x) = 6 \end{cases}$$

$$p^{(IV)}(x) = 0$$

$$p^{(V)}(x) = 0$$

$$\boxed{p^{(n)}(x) = 0 \quad , \quad n \geq 4}$$

$f = \sin x$

$f' = \frac{d}{dx}(\sin x) = \cos x$

$f = \sin x$

$$f' = \frac{d}{dx}(\sin x) = \underline{\cos x}$$

$$f'' = \frac{d}{dx}(\cos x) = -\sin x$$

$$\boxed{f = -f''}$$