Föreläsning 3.2
den 14 september 2020 12:56

Mål §2.6 & 2.7 – Complete some info. from last class.
§2.8 – Medelvärdessatsen, växande och avtagande funktioner
Rolles sats
§3.1 – Inversa funktioner och deras derivator

§2.6.

$f(x)$

$f'(x) = \underbrace{\frac{d}{dx}} f(x)$ (first derivative)

$\underbrace{f''(x)} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = \frac{d^2}{dx^2} f(x) \leftarrow$

$f'''(x) = \underbrace{\frac{d}{dx}\left(f''(x)\right)} = \underbrace{\frac{d^3}{dx^3} f(x)}$ ... and so on.

Obs $f(x) = P_n(x)$ polynomial with degree n

$\underbrace{f(x) = a_n x^n} + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

let's think about the simplest case:

$\boxed{f(x) = x^n}$, $n \in \mathbb{N}$.

$f'(x) = n\, x^{n-1}$

$f''(x) = n.(n-1)\, x^{n-2}$

$f'''(x) = n(n-1)(n-2)\, x^{n-3}$

$\vdots$

$f^{(n-1)}(x) = n(n-1)(n-2)\dots 2.x^{n-(n-1)}$ $\quad n - \cancel{n} + 1 = 1$

$\boxed{f^{(n)}(x) = n!\, x^0 = n!} = (n)(n-1)(n-2)\dots 2.1.$

$\underbrace{f^{(k)}(x),\ k > n} \Rightarrow \underbrace{f^{(k)}(x) \equiv 0}$ (identically equal zero)

Example 3 page 129.

If [A, B, K are constants] Then [$\boxed{y = A\cos(kt) + B\sin(kt)}$ is a solution of the 2^{nd} order differential equation (known as differential equation (known as the differental equation of Simpel Harmonic Motion)

$$\boxed{\frac{d^2}{dt^2}y(t) + K\,y(t) = 0}$$

Proof Obs. To verify that a function is a solution of a differential equation means that the function satisfies the restrictions stated by the equation. So, substituting the function expressio on the equation, the balance is preserved.
(Left side = Right side)

$y(t) = A\cos(k.t) + B\sin kt$ ⊠

$\boxed{y'' + Ky = 0}$

$y'(t) = A.k(-\sin kt) + B.k\cos(k.t)$

$y''(t) = Ak^2(-\cos kt) + B.k.k(-\sin(kt))$

$y''(t) = -Ak^2\cos(kt) - Bk^2\sin(kt)$ (*)

Now we substitute the expression y, y' and y'' in the differential equation: $\underbrace{\frac{d^2}{dt^2}y(t)}_{(*)} + k^2 \underbrace{y(t)}_{⊠} = 0$

$$\boxed{-Ak^2\cos kt - Bk^2\sin kt} +$$

$$k^2(A\cos kt + B\sin kt) =$$

$$\cos kt\,(-Ak^2 + k^2A) + \sin kt(-Bk^2 + Bk^2) =$$

$$\cos kt \cdot (0) + \sin(kt)(0) = 0$$

Therefore $y(t)$ satisfies the equation and we say that $y(t)$ is a solution of the differential equation.

Obs: In a future Math Course you are going to investigate

If this is the only type of function that satisfies the equation or not.

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§2.7 Page 131 (Please, read carefully this Section)

If one quantity y depends on another quantity x $(y = f(x))$,

we want to know:

How a change in the value of x by an amount Δx will affect the value of y.

So $\Delta y = f(x+\Delta x) - f(x)$ is the variation.

If Δx is small, then we can get a better approximation for Δy, since $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ when Δx is small.

thus $$\boxed{\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x \approx \frac{dy}{dx} \cdot \Delta x = f'(x)\,\Delta x}$$

When we represent these approximations in terms of differentials we have: $(\Delta x \to dx\ ,\ \Delta y \to dy)$ then

$$\Delta y \approx \boxed{dy = f'(x)\,dx} \quad \text{(differentials)}$$

$$\boxed{\frac{dy}{dx} = f'(x)} \quad \text{(by definition)}$$

$$\frac{\Delta y}{\Delta x}$$

§2.8 The Mean-Value Theorem

1) Max-Min Theo.

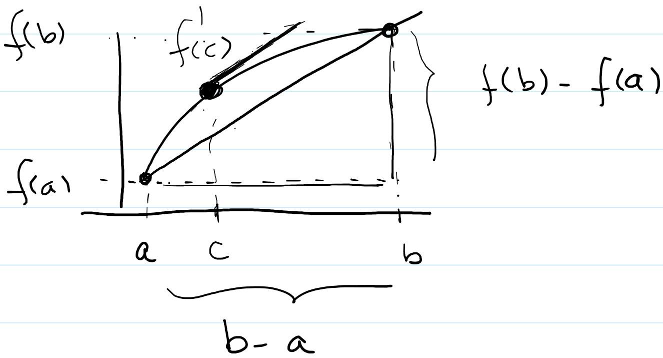
$\left[\ \underline{I = [a,b]}\ \text{limited \& closed interval}\ \right] \Rightarrow \left[\ \exists\ p, q \in (a,b)\ /\ \right.$

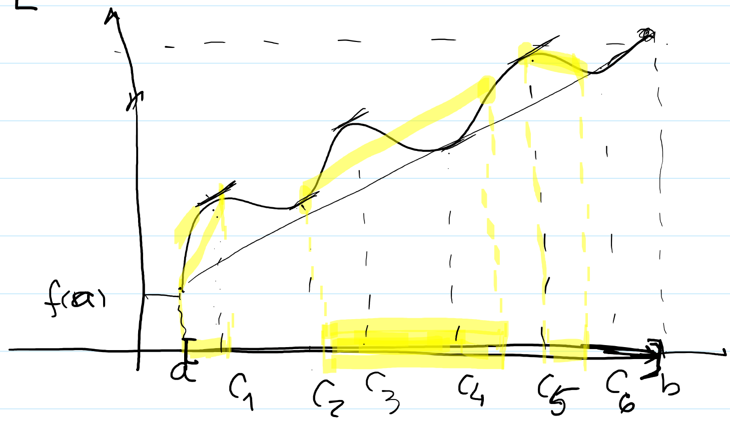
$f(p) \le f(x) \le f(q) \quad \forall x \in I$

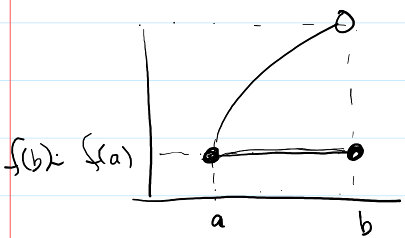
$$\left[\begin{array}{l} I=[a,b] \text{ limited \& closed interval} \\ f: I \to \mathbb{R} \quad \text{continuous} \end{array}\right] \Rightarrow \left[\begin{array}{l} \exists\, p,q \in (a,b) \;/ \\ \underset{\min}{f(p)} \leq f(x) \leq \underbrace{f(q)}_{\text{Max}} ,\ \forall x \in I \end{array}\right]$$

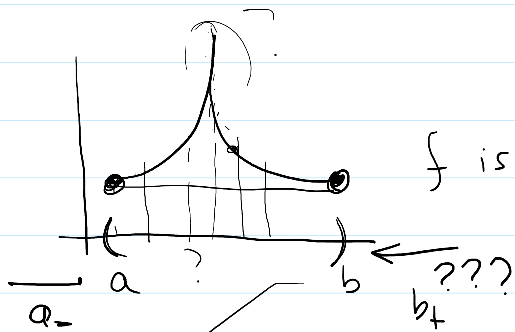
2) pag 138 Theo 11. Mean-Value Theo.

$$\left[\begin{array}{l} I=[a,b] \text{ closed \& limited} \\ f:[a,b] \to \mathbb{R} \\ f \text{ is continuous on } I \\ f \text{ is differentiable on } (a,b) \end{array}\right] \Rightarrow \left[\begin{array}{l} \exists\, c \in (a,b) \;/ \text{ such that} \\ \dfrac{f(b)-f(a)}{b-a} = f'(c) \end{array}\right]$$







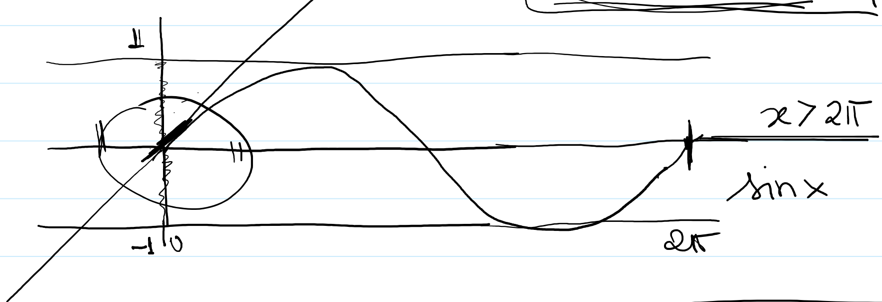


f is not diff.

Application of The Mean-Value-Theorem

Ex2 Show that $\boxed{\sin x < x}$, $\forall x > 0$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



1) If $x > 2\pi$ then $\sin x \leq 1 < 2\pi < x$

2) If $0 < x < 2\pi$ $\qquad \underbrace{I = [0, 2\pi]}$ $\quad \underbrace{I^* = (0, 2\pi)}$

Hypo
$$\begin{cases} f(t) = \sin t \text{ is well defined on } I. \\ f(t) = \sin t \text{ is continuous on } I \\ f(t) \text{ is differentiable on } I^* = (0, 2\pi) \end{cases}$$

So by the Mean-Value Theorem $\qquad \underline{\underline{(0,x)}} \subseteq I$

$\exists\, \underline{\underline{c}} \in (0,x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \qquad\qquad f'(t) = \cos t$$
$$f'(c) = \cos c$$

$$\frac{\sin x}{x} = \frac{\sin x - \cancel{\sin 0}}{x - \cancel{0}} = \cos(c) < \underline{1} \quad \forall c \in (0,x) \subseteq I = [0, 2\pi]$$

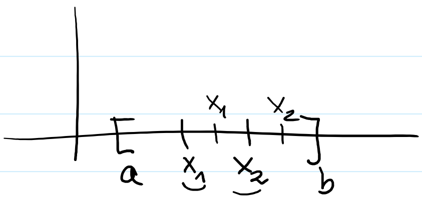
$$\Rightarrow \quad \frac{\sin x}{x} < 1 \Rightarrow \boxed{\sin x < x} \quad \forall x \in (0, 2\pi)$$

$$\sin(x) < x \quad , \forall x \geq 2\pi$$

$$\boxed{\sin x < x} \quad \forall x > 0. \qquad \square$$

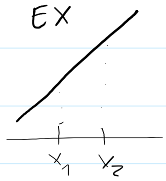
Page 140 $\quad$ Def 6

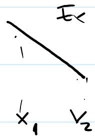
$f: [a,b] \to \mathbb{R}$



$x_1 < x_2$

If	Then
$f(x_2) > f(x_1)$, $\forall x_2 > x_1$	f is increasing on I
$f(x_2) < f(x_1)$, $\forall x_2 > x_1$	f is Decreasing on I
$f(x_2) \geq f(x_1)$ $\forall x_2 > x_1$	f is Non-decreasing on I
$f(x_2) \leq f(x_1)$ $\forall x_2 > x_1$	f is Non-increasing ...

EX


Ex


$f(x_2) \leq f(x_1) \quad \forall\, x_2 > x_1$ f is Non-increasing on I

$\boxed{x_2 > x_1 \quad , x_2\, x_1 \in}$ $I = [a,b]$

$f : [a,b] \to \mathbb{R}$ ✓

f is continuous on $[a,b]$

f is diff on (a,b)

$\boxed{x_2 > x_1} \iff \boxed{(x_2 - x_1) > 0}$

By the Mean-Value Theorem

$\exists\, c \;,\; \boxed{x_1} < c < \boxed{x_2}$ $\quad c \in (x_1, x_2) \subset (a,b)$

$$\boxed{\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)} \Rightarrow$$

$$f(x_2) - f(x_1) = f'(c)\, \overbrace{(x_2 - x_1)}^{>0}$$

if $f'(c) > 0$ then $f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1)$

f is increasing on $(x_1, x_2) \subset (a,b)$

if $f'(c) < 0$ then

$f(x_2) - f(x_1) < 0$

$\Rightarrow$ f is decreasing on I.

if $\boxed{f'(c) \geq 0} \Rightarrow \boxed{f(x_2) \geq f(x_1)} \Rightarrow$ f is Non-decreasing

if $f'(c) \leq 0 \Rightarrow f(x_2) \leq f(x_1) \Rightarrow$ f is Non-increasing

Theorem 12 on page 141

Remark of Theorem 12.

Obs: [$f'(x_0) > 0$ at a single x_0 DOES NOT imply that f is increasing on any Interval containing x_0

Exercise 30 pag 145

Let $f(x) = \begin{cases} x + 2x^2 \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(a) Show that $f'(0) = 1$ (Hint: use definition of derivative)

(b) Show that ANY interval containing $x=0$ contains points where $f'(x) < 0$, so f cannot be increasing on such intervals. (f is an oscillating function near zero!)

(a) $f'(0)$ by definition:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} =$$

$$\lim_{h\to 0} \frac{\left((0+h) + 2(0+h)^2 \sin\left(\frac{1}{0+h}\right)\right) - 0}{h} =$$

$$\lim_{h\to 0} \frac{h}{h} + \frac{2h^{\cancel{2}}}{\cancel{h}} \sin\left(\frac{1}{h}\right) =$$

$$\underbrace{\lim_{h\to 0} 1}_{\substack{\| \\ 1}} + \lim_{h\to 0} 2h \sin\left(\frac{1}{h}\right) = \left[2\cdot 0 \cdot \sin\left(\frac{1}{0}\right)\right] ???$$

we can not solve it directly

Because the function is constant

Since $|\sin t| \leq 1$, $\forall t \in \mathbb{R}$

$-1 \leq \sin t \leq 1$

So

$$-2h \leq 2h \sin\left(\frac{1}{h}\right) \leq 2h$$

By the Sequeze Theorem

$\lim_{h\to 0} -2h = 0$ & $\lim_{h\to 0} 2h = 0$

since $2h\sin\left(\frac{1}{h}\right)$ is bounded by these two functions

$$\lim_{h\to} 2h\sin\left(\frac{1}{h}\right) = 0 = \lim_{h\to 0} -2h = \lim_{h\to 0} 2h$$

So $\underbrace{f'(0) = 1 + 0 = 1}$ as we wanted.

$\square$

b) Now we want to find points so close to zero as possible, whose function slopes are negative

b.1) $f'(x)$ computed by the rules on $(a,b) = (0,1)$

$$f(x) = x + 2x^2 \cdot \sin\left(\frac{1}{x}\right)$$

$$f'(x) = 1 + (2 \cdot 2x)\sin\frac{1}{x} + (2x^2) \cdot \left(\cos\left(\frac{1}{x}\right) \cdot (-x^{-2})\right)$$

$$f'(x) = 1 + 4x\sin\left(\frac{1}{x}\right) - 2\cancel{x^2} \cdot \frac{1}{\cancel{x^2}} \cdot \cos\left(\frac{1}{x}\right)$$

$$\boxed{f'(x) = 1 + 4x\sin\left(\frac{1}{x}\right) - 2\cos\left(\frac{1}{x}\right)}$$

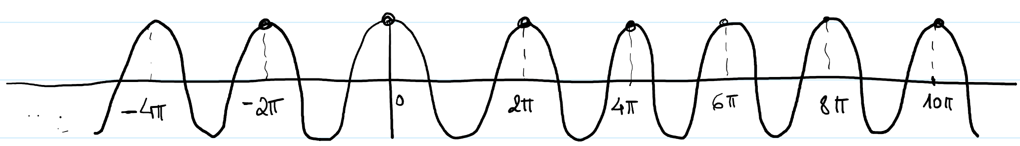
This is the function that oscillates near $x=0$.

The oscillation is caused by $\sin\left(\frac{1}{x}\right)$ & $\cos\left(\frac{1}{x}\right)$.

Now let's find values near zero / $f'(x) > 0$ & $f'(x) < 0$

1) $\cos(2\pi) = 1 \qquad \cos(2\pi + k \cdot 2\pi) = 1 \qquad k \in \mathbb{Z}$

$\qquad \cos((k+1)2\pi) = 1 \quad , k \in \mathbb{Z}.$



But our function is $\cos\left(\frac{1}{x}\right)$

$\frac{1}{x} = (k+1)2\pi \quad k \in \mathbb{Z} \iff x = \frac{1}{\quad} \quad , k \in \mathbb{Z}.$

so $\frac{1}{x} = (k+1)2\pi$, $k \in \mathbb{Z}$ $\iff$ $x = \frac{1}{(k+1)2\pi}$, $k \in \mathbb{Z}$.

Obs $a_k = (k+1)2\pi \xrightarrow[k\to+\infty]{} +\infty$ so $x_k = \frac{1}{a_k} \xrightarrow[k\to+\infty]{} 0$

$x_k = \frac{1}{(k+1)2\pi}$, $k \in \mathbb{Z}$

$$\cos\left(\frac{1}{x_k}\right) = \cos\left(\frac{1}{\frac{1}{(k+1)2\pi}}\right) = \cos((k+1)2\pi) = 1, \ \forall k \in \mathbb{Z}$$

In this case $x_k \xrightarrow[k\to\infty]{} 0 \implies \cos\left(\frac{1}{x_k}\right) = 1$

In this case

$$f'(x_k) = 1 + 4x_k \sin\left(\frac{1}{x_k}\right) - 2\cos\left(\frac{1}{x_k}\right)$$

$$f'(x_k) = 1 + 4\frac{1}{(k+1)2\pi}\cancel{\sin(2\pi(k+1))}_{0} - 2\underbrace{\cos(2\pi(k+1))}_{=1}$$

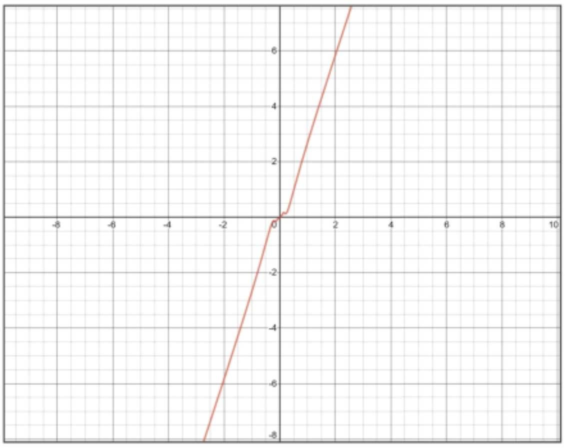
$$f'(x_k) = 1 - 2 = -1$$

Here we have $\boxed{f'(0) = 1 > 0}$

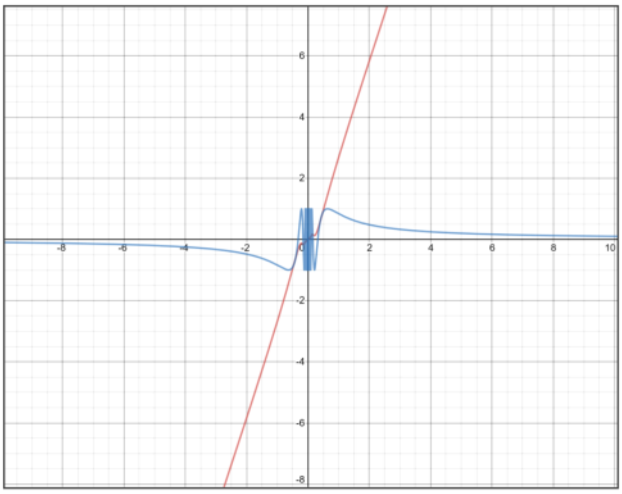
And we constructed a sequence of values so close to zero as we want, whose function slopes are negative $(= -1)$

$x_k = \frac{1}{(k+1)2\pi}$ such that $f'(x_k) = -1 < 0$

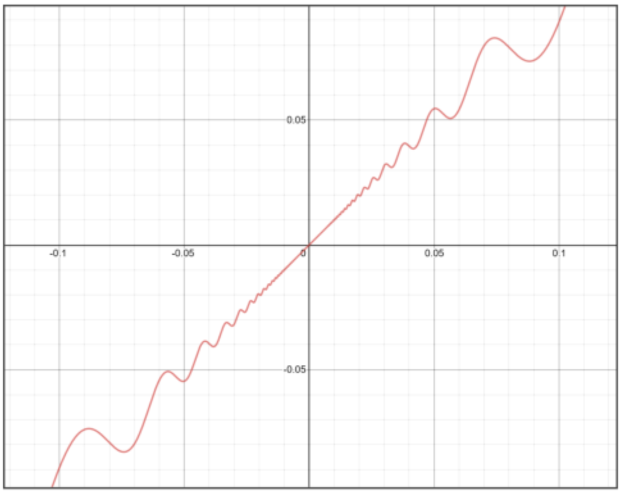
fx



Here is the graph of
$f(x)$.
when the window of
representation is
to wide, we
do not see the
ocillations



Here is to
show that
$\sin\left(\frac{1}{x}\right)$ is realy
oscilating ner 0;
So $f(x)$ carries
also oscillations
near zero.



Here is
a zoom to
the function