Mål :
- Avsluta § 2.8
- § 3.1 - inversa funktioner
- ln , exp
- inversa trigonometriska funktioner

Exempel 4 page 141

$f(x) = x^3 - 12x + 1$. On what intervals is $f(x)$ increasing?
On what intervals is $f(x)$ decreasing?

Solution.

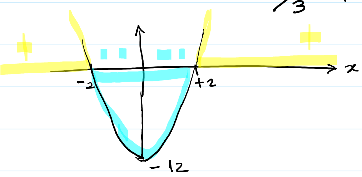
* Knowing where $f'(x) > 0$, we find where f is increasing.
* Knowing where $f'(x) < 0$, we find where f is decreasing.

$f(x) = x^3 - 12x + 1$

$f'(x) = 3x^2 - 12$ $\quad f'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow$

$x^2 = 12/3 = 4 \Rightarrow x = \pm 2$

$f'(x) = 3(x-2)(x+2)$



So

$-2 < x < 2 \Rightarrow f'(x) < 0 \Rightarrow f(x)$ is decreasing on

$I_1 = (-2, 2)$

$(-\infty, -2) \cup (2, +\infty) \Rightarrow f'(x) > 0 \Rightarrow f(x)$ is increasing on

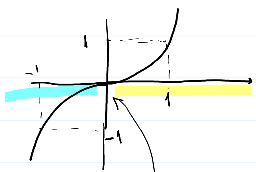
$I_2 = (-\infty, -2) \cup I_3$

$I_3 = (2, +\infty)$

Example 5

A function f whose derivative satisfies
$f'(x) \geq 0$ on an Interval , can still be increasing,
rather then just non-decreasing as theo 12(c) stated.
This will happen if $f'(x) = 0$ ONLY at isolated points,
so that f is assumed to be increasing on intervals
to the left and to the right of this point.

$f(x) = x^3$ is one example of that



$f'(x) = 3x^2 \geq 0 \quad \forall x \in \mathbb{R}$

$f'(x) = 0 \Leftrightarrow 3x^2 = 0 \Leftrightarrow x = 0$

so $f'(0) = 0$ BUT $\rightarrow f'(x) > 0,\ x > 0$
& $\rightarrow f'(x) > 0,\ x < 0$

$f'(x) = 0$
only at $x = 0$,
being an isolated event

Theorem 13 pag 142

$\left[\begin{array}{l} I = [a,b] \text{ finite \& closed interval} \\ f: [a,b] \to \mathbb{R} \text{ continuous} \end{array}\right]$. $\left[\quad f(x) = \text{cte} \quad \forall x \in [a,b] \right]$

$$\left[\begin{array}{l} I=[a,b] \text{ finite \& closed interval} \\ f:[a,b]\to\mathbb{R} \text{ continuous} \\ f:(a,b)\to\mathbb{R} \text{ differentiable} \\ f'(x)=0,\ \forall x\in(a,b) \end{array}\right] \Rightarrow \left[\begin{array}{l} f(x)\equiv cte,\ \forall x\in[a,b] \\ f \text{ is constant on } I. \end{array}\right]$$

Proof (Using the Mean-Value Theorem)

pick a point $x_0 \in I$ $(a < x_0 < b)$

$f(x_0) = \text{value} = C_1$

Chose $x \neq x_0$ $x \in I$

So applying the Mean-Value-Theorem for $I^* = [x, x_0] \subset I = [a,b]$

there exists $c \in (x, x_0)$ $(x < c < x_0)$ such that

$$\frac{f(x)-f(x_0)}{x-x_0} = \boxed{f'(c)}$$

But, by hypothesis $f'(x)=0$, $\forall x \in (a,b)$ $\therefore$
therefore $f'(c) = 0$ too.

So, $\dfrac{f(x)-f(x_0)}{x-x_0} = 0 \Rightarrow f(x)-f(x_0) = 0 \Rightarrow$

$$f(x) = f(x_0) = C_1, \quad \forall x \in (x_0, x) \subset (a,b) \subset [a,b].$$

So f is constant on the entire domain.

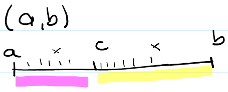
Theorem 14 pag 142

Observe that $(\not\Leftarrow)$

$$\left[\begin{array}{l} (a,b) \text{ finite oppen interval} \\ f:(a,b)\to\mathbb{R} \\ f \text{ achieves Max OR min} \\ \text{value at } c\in(a,b) \\ f'(c) \text{ exists} \end{array}\right] \Rightarrow \left[\begin{array}{l} \boxed{f'(c) = 0} \\ c \text{ is called critical point of } f \end{array}\right]$$

Proof

Suppose that f has a Max value on (a,b)
at point $x = c$, $c \in (a,b)$.
So $f(c) \geq f(x)$, $\forall x \in (a,b)$



If $c < x < b$ then $\dfrac{f(x)-f(c)}{x-c} \leq 0$ $\quad \leq 0 \quad > 0$

So $f'(c) = \displaystyle\lim_{x \to c_+} \frac{f(x)-f(c)}{x-c} \leq 0$

Now, let's assume $a < x < c$, then

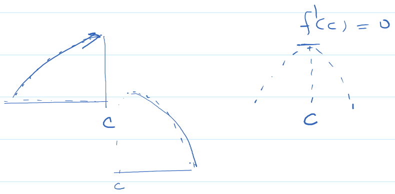
$$\frac{f(c)-f(x)}{c-x} \geq 0$$

So $f'(c) = \lim_{x \to c_-} \frac{f(c) - f(x)}{c - x} \geq 0$

In this way we have:

$f'(c) \geq 0 \qquad x \to c_-$

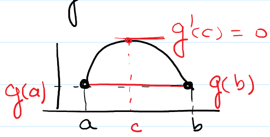
$f'(c) \leq 0 \qquad x \to c_+$



Thus $\boxed{f'(c) = 0}$ □

Theorem 15 (Rolle's theorem (particular case of the Mean-Value Theorem))

$\left[\begin{array}{l} [a,b] \text{ finite, closed interval} \\ g : [a,b] \to \mathbb{R} \text{ defined} \\ g \text{ continuous on } I = [a,b] \\ g \text{ differentiable on } (a,b) \\ g(a) = g(b) \end{array}\right] \Longrightarrow \left[\begin{array}{l} \text{There exist } c \in (a,b) \\ \text{such that} \\ \qquad g'(c) = 0 \end{array}\right]$



Proof

CASE 1: If $g(x) = g(a) = g(b)$ $\forall x \in [a,b]$ then $g(x) \equiv cte$ so $g'(c) = 0$, $\forall c \in (a,b)$.

CASE 2 Suppose $g(x) \neq$ constant function. Therefore, there must exist $x \in (a,b)$ such that $g(x) \neq g(a)$ $\quad g(x) \neq g(b)$.

Let's assume that $g(x) > g(a)$

Since g is continuous on $[a,b]$, by the Min-Max theorem $\exists\, c \in (a,b)$ such that

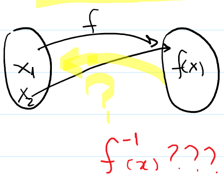
$\text{Max} = g(c) \geq g(x) \quad, \forall x \in (a,b)$

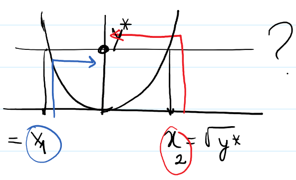
So $\quad g(c) \geq g(x) > g(a) = g(b) \quad$ (so $c \neq a$, $c \neq b$)

By the hypothesis g is a differentiable function, so $g'(c) = 0$ by theorem 14. (c is a critical point). □

So the result is obtained. □

A §3.1 Inversa funktioner och deras Derivator



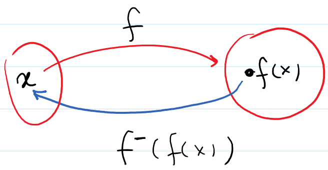


In order to be able to compute f^{-1} (the inverse of f), the function f itself has to be an one-to-one function, which means:

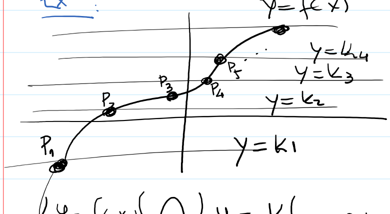
a) $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or

b) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$



Ex:



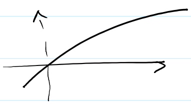
$\{y=f(x)\} \cap \{y=k\}$ = only one point

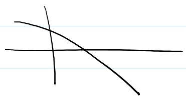
f is one-to-one (1:1)
(an injektion)

if $\{y=f(x)\} \cap \{y=k,\ k\in\mathbb{Z}\}$

is just one point on $y=f(x)$

2) f increasing function or f decreasing function $\Rightarrow f$ 1:1
(one-to-one)





Definition 2

[f is one-to-one] $\Rightarrow$ [f has an inverse function f^{-1}. The value $f^{-1}(x)$ is the unique number $y \in D_f \equiv$ domain of f for which $f(y)=x$]

$$\boxed{y = f^{-1}(x) \iff x = f(y)}$$

$$\boxed{f(f^{-1}(x)) = x}$$

Example.

Show that $f(x) = 2x-1$ is 1:1 and find f^{-1}

a) $f(x)=2x-1$ is 1:1

i) Assume that $\boxed{f(x_1)=f(x_2)} \in V_f = \text{Range}(f)$

So $2x_1 - \cancel{1} = 2x_2 - \cancel{1} \Rightarrow$

$\cancel{2}x_1 = \cancel{2}x_2 \Rightarrow \boxed{x_1 = x_2}$

Obs: We could also show that $f(x)$ is an increasing function

$f(x) = 2x-1 \Rightarrow f'(x) = 2 > 0,\ \forall x \in D_f$.

So f is increasing and therefore f is 1:1.

b) Let's compute f^{-1}

Remember $y = f^{-1}(x) \iff \underbrace{f(y) = x}_{*}$

$x = 2y - 1$ Now we isolate y to obtain the expression of $f^{-1}(x)$

$\downarrow$

$x + 1 = 2y \Rightarrow \boxed{y = \frac{x+1}{2}}$ $y = f^{-1}(x)$

——/——

Properties of the Inverse functions.

1) $y = f^{-1}(x) \Leftrightarrow x = f(y)$

2) The Domain of f^{-1} = Range of f

3) The Range of f^{-1} = Domain of f

4) $f^{-1}(f(x)) = x \quad \forall x \in D_f$.

5) $f(f^{-1}(x)) = x, \quad \forall x \in D_{f^{-1}}$

6) $(f^{-1})^{-1}(x) = f(x) \quad \forall x \in D_f$

7) The graph of f^{-1} is a reflection of the graph of f with respect to the line $y = x$

Ex2 $g(x) = \sqrt{2x+1}$

1) Show that g is invertible
2) Compute its inverse
3) Draw g^{-1} graph

1) g is 1:1 ?

$g(x_1) = g(x_2) \Rightarrow x_1 = x_2$?

Let's see:

$\sqrt{2x_1+1} = \sqrt{2x_2+1} \Rightarrow 2x_1 + \cancel{1} = 2x_2 + \cancel{1} \Rightarrow \cancel{2}x_1 = \cancel{2}x_2$

$\Rightarrow x_1 = x_2$ ✓ yes! $g(x)$ is one-to-one.

2) To compute $g^{-1}(x)$. $\boxed{x = g(y)} \Leftrightarrow g^{-1}(x) = y$

$x = \sqrt{2y+1} \Rightarrow x^2 = 2y + 1 \Rightarrow x^2 - 1 = 2y \Rightarrow$

$\Rightarrow y = \frac{x^2-1}{2}$

$g^{-1}: D_{g^{-1}} \longrightarrow R_{g^{-1}}$

$x \longmapsto g^{-1}(x) = \frac{x^2-1}{2}$

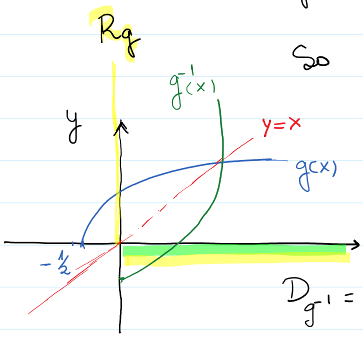
But $D_{g^{-1}} = R_g$

$g(x) = \sqrt{2x+1}$ $\quad g(x) \in [0, +\infty) = \mathbb{R}_+$

R_g

So $g^{-1}: \mathbb{R}_+ \longrightarrow [-\frac{1}{2}, +\infty)$

$g(x) = \sqrt{2x+1}$ $\quad g(x) \in [0, +\infty) = \mathbb{R}_+$



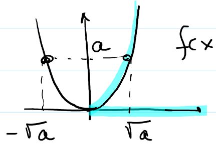
So $\quad g^{-1} : \mathbb{R}_+ \longrightarrow [-\frac{1}{2}, +\infty)$

$\qquad\qquad x \longrightarrow \dfrac{x^2-1}{2}$

$D_{g^{-1}} = R_g$

Obs 1 $\quad f(x) = \dfrac{1}{x} \Rightarrow f^{-1}(x) = f(x) = \dfrac{1}{x}$ $\quad$ example of self-inverse function.

Obs 2 :



$f(x) = x^2$

$\left[\begin{array}{l} f : \mathbb{R} \to \mathbb{R} \\ f \text{ is NOT } 1:1 \end{array}\right.$

BUT

$\left\{\begin{array}{l} f : \mathbb{R}_+ \longrightarrow \mathbb{R} \\ \text{is } 1:1 \end{array}\right.$

& f is now invertible.

Derivatives of Inverse functions

Hypothesis :
$\left[\begin{array}{l} f : (a,b) \to \mathbb{R} \text{ differentiable} \\ f \text{ is one-to-one on } (a,b) \\ f \text{ is invertible on } (a,b) \\ \text{So } \exists f^{-1} \end{array}\right]$

Goal : We want to compute $\dfrac{d}{dx}(f^{-1})$.

Consider :

$f(f^{-1}(x)) = x$ $\quad$ (for all $x \in D_{f^{-1}}$) $\quad$ (property 5)

So $\quad \underbrace{\dfrac{d}{dx}\left(f(f^{-1}(x))\right)}_{\text{using the chain Rule}} = \dfrac{d}{dx} x = 1$

$f'(f^{-1}(x)) \cdot \underbrace{\dfrac{d}{dx}(f^{-1}(x))}_{\text{isolating this term}} = 1$

$$\boxed{\dfrac{d}{dx}(f^{-1}(x)) = \dfrac{1}{f'(f^{-1}(x))}}$$

Exampel 4 page 170

1) Show that $f(x) = x^3 + x$ is one-to-one on $\mathbb{R}$.

2) Obtain $f^{-1}(x)$ & $(f^{-1})' = \frac{d}{dx}\left(f^{-1}(x)\right)$

3) Knowing that $f(2) = 10$, compute $(f^{-1})'(10)$.

Solution

1) $f(x) = x^3 + x$ $\quad \frac{d}{dx} f(x) = 3x^2 + 1 > 0, \ \forall x \in \mathbb{R} \Rightarrow f$ is one-to-on, $\forall x \in \mathbb{R}$

So f is invertible on $\mathbb{R}$.

2) To compute $(f^{-1})'(x) = \frac{d}{dx}\left(f^{-1}(x)\right)$

we have to come back to property (5)

$f(f^{-1}(x)) = x$ (considering the chain Rule)

$$\frac{d}{dx} f\left(f^{-1}(x)\right) \cdot \left(\frac{d}{dx} f^{-1}(x)\right) = 1 \Rightarrow$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Now $\quad f(x) = x^3 + x$

1) Starting point to compute $\quad \boxed{y = f^{-1}(x) \iff x = f(y)}$

$\boxed{x = y^3 + y}$ property (5)

2) Now that we wrote using property 5, we can derivate by the chain Rule.

$$\frac{d}{dx} x = \underbrace{f'(\underbrace{f^{-1}(x)}_{y})}\cdot \frac{d}{dx} f^{-1}(x)$$

$$1 = f'(y) \cdot \frac{d}{dx} f^{-1}(x)$$

$$1 = (3y^2 + 1) \cdot \frac{d}{dx} f^{-1}(x)$$

So $\quad \boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{3y^2 + 1}}$

3) Now $\quad x = f(2) = 10 \Rightarrow y = f^{-1}(10) = f^{-1}(f(2)) = 2$

Thus $\quad (f^{-1})'(10) = \left.\frac{1}{3y^2 + 1}\right|_{y=2} = \frac{1}{3(4) + 1} = \frac{1}{13}$