Mål
- Några viktiga Resultat från F. 3.3 om exp funktion
- Kritiska punkter, test med f' & f''
- Konvexitet / Konkavitet
- Inflektionspunkter
- Intervallhalvering / Bissection method

Natural Logarithm $\ln x = \log_e x$

The exponential function $\exp x = e^x$

Def 6 pag 176

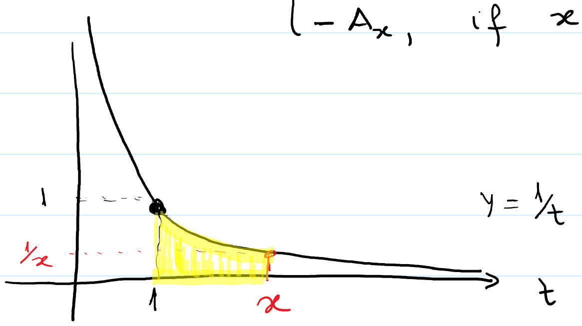
For $x>0$, let A_x be the area of the plane region bounded by the curve $y = 1/t$, the t-axis and the vertical lines $t=1$, $t=x$.

The function $y = \ln x$ is defined by:

$$\ln(x) = \begin{cases} A_x, & \text{if } x>1 \\ -A_x, & \text{if } x<1 \end{cases} \Rightarrow \begin{cases} \ln x > 0, & x>1 \\ \ln x < 0, & 0<x<1 \end{cases}$$

$\Downarrow$

$\ln(1) = 0.$



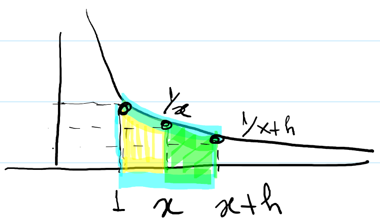
D_f: $x \in \mathbb{R}_+ \setminus \{0\}$

$x \in (0, +\infty)$

Theorem 1 page 177

$$\boxed{\text{If } x>0, \text{ then } \frac{d}{dx}\ln(x) = \frac{1}{x}}$$

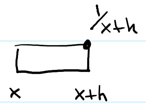
Proof: case 1: $\ln(x) < \ln(x+h)$

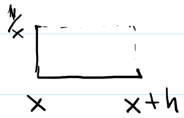


Area (x, x+h) $= \ln(x+h) - \ln(x)$

Comparing the areas with the rectangles, we have:

rectangles, we have:



$< \ln(x+h) - \ln(x) <$


$$\frac{h \cdot \frac{1}{x+h}}{h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{h \cdot \frac{1}{x}}{h}$$

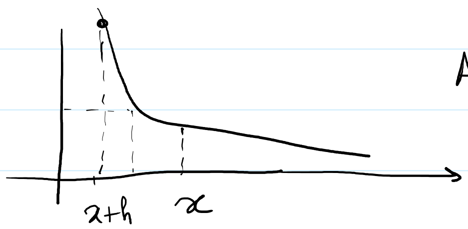
Newton Quotient

$$\lim_{h\to 0} \frac{1}{x+h} < \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} < \lim_{h\to 0} \frac{1}{x}$$

By the squeeze-theorem, we have:

$$\frac{1}{x} \leq \frac{d}{dx}\left(\ln(x)\right) \leq \frac{1}{x} \Rightarrow \boxed{\frac{d}{dx}\ln(x) = \frac{1}{x}}$$

Case 2 : $\ln(x+h) < \ln(x)$



Analogous.

Conclusion: :

$$\begin{cases} \frac{d}{dx}\ln x = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x} \\ \ln(1) = 0. \end{cases}$$

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a) $\lim_{x\to +\infty} \ln(x) = +\infty$

b) $\lim_{x\to 0_+} \ln(x) = -\infty$

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1) $\frac{d}{dx}\ln|x| = \frac{1}{x}$, $\forall x \neq 0$

2) $\int \frac{1}{x}\,dx = \ln|x| + C$ a) $\int f(x)\,dx$ means the inverse

$\frac{d}{dx}$ x

2) $\int \frac{1}{x}\,dx = \underbrace{\ln|x| + c}$

•) $\int f(x)\,dx$ means the inverse operation with respect to the derivation.

obs $\left(\int f(x)\,dx = F(x) + C \text{ if } \frac{d}{dx}\left(F(x)+C\right) = f(x)\right)$

•) C is called constant of integration since we have $\frac{d}{dx}(F(x)+C) = \frac{d}{dx}(F(x))$

<u>Proof</u>:

Theor. 1 page 177

1) If $x>0$, then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x \overset{*}{=} \frac{1}{x}$

theo 1 + Chain Rule.

If $x<0$ then $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{(-x)}\cdot(-1) = \frac{1}{x}$

2) On any interval not containing zero, we have:

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{Because} \quad \frac{d}{dx}\left(\ln|x|+c\right) = \frac{1}{x} + 0 = \frac{1}{x}$$

<u>Examples</u>: Derivate:

a) $\frac{d}{dx}\left(\ln|\cos x|\right)$ b) $\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right)$

Solution

Theo 1 + Chain Rule

a) $\frac{d}{dx}\left(\ln|\cos x|\right) = \frac{1}{\cos x}\cdot\frac{d}{dx}\cos x = \frac{1}{\cos x}(-\sin x) = -\tan x$

Theo 1

b) $\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1}{(x+\sqrt{x^2+1})}\cdot\left(1 + \frac{1}{\cancel{2}}(x^2+1)^{-\frac{1}{2}}\cdot\cancel{2}x\right)$

$\frac{1}{(x+\sqrt{x^2+1})}\cdot\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$

$$\frac{1}{\cancel{(x+\sqrt{x^2+1})}} \cdot \left(\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}} \right) =$$

$$= \frac{1}{(\sqrt{x^2+1})}$$

The function $\ln x$ is one-to-one in $D_{\ln(x)} = (0,+\infty)$

So $\ln x$ is invertible on $D_{\ln(x)} = (0,+\infty)$

Def Let's call the inverse of $\ln(x)$ as the function $\exp x$

$$y = \exp x \iff \ln(y) = x \quad , \quad y > 0.$$

Since $\ln(1) = 0 \Rightarrow \exp(0) = 1.$

$$D_{\exp(x)} = \text{Range}_{\ln(x)} = (-\infty,+\infty)$$

So
$$\begin{cases} \ln(\exp(x)) = x \; , \; \forall x \in \mathbb{R} \\ \exp(\ln(x)) = x \; ; \; \forall x > 0 \end{cases}$$

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Let $e = \exp(1) \iff \ln(e) = 1$

$e^x = \exp x.$

1) e is a irrational number

$e \neq p/q \; , \; q \neq 0 \; , \; p, q \in \mathbb{Z}.$

e is not a zero of any polynomial with rational coefficients

2) $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}$

3) $\lim_{x \to +\infty} e^x = +\infty$

4) $\lim_{x \to -\infty} e^x = 0$

<u>Notation</u>: $\exp x = e^x, \quad x \in \mathbb{R}$

$\log_e x = \ln x, \quad x \in (0, +\infty)$

Derivative of e^x and a^x, $a > 1$ $a \in \mathbb{R}$

By implicit derivation, we have:

$$y = e^x \Rightarrow x = \ln y \Rightarrow$$

$$\boxed{1 = \frac{dx}{dx} = \frac{d \ln y}{dy} \cdot \frac{dy}{dx}} \quad \text{Implicit Derivation} \Rightarrow$$

$$1 = \frac{1}{y} \cdot \frac{dy}{dx} \Rightarrow$$

$$\boxed{y = \frac{dy}{dx}} \Longrightarrow \boxed{\frac{d}{dx} e^x = e^x}$$

Def 7: $a^x = e^{\ln(a^x)} = e^{x \ln(a)}$ $\quad a \neq 1,\ a > 0 \quad x \in \mathbb{R}$

So $\frac{d}{dx} a^x = \frac{d}{dx} \underbrace{e^{x \ln(a)}}_{} = \underbrace{e^{x \ln(a)}}_{} \cdot \ln(a) =$

chain Rule

$$\boxed{\frac{da^x}{dx} = a^x \cdot \ln(a)} \quad \begin{cases} < 0, & \text{if } 0 < a < 1 \\ > 0, & \text{if } a > 1 \end{cases} \quad \circledast$$

<u>Ex 2</u> $\frac{d}{dx}(x^a) = \frac{d}{dx}\left(e^{\ln x^a}\right) = \frac{d}{dx} e^{a \ln x}$ $\underline{\underline{\text{Chain Rule}}}$

Ex 2 $\quad \frac{d}{dx}(x^a) = \frac{d}{dx}\left(e^{a\ln x}\right) = \frac{d}{dx} e^{a\ln x}$ Chain Rule

$$= \underbrace{e^{a\ln x}} \cdot a\frac{1}{x} = x^a \cdot a \cdot \frac{1}{x} = \underbrace{a x^{a-1}}$$

$$\boxed{\frac{d x^a}{dx} = a x^{a-1}}$$

⊛ Thus $y = a^x$ is 1:1

So $y = a^x$ has an inverse function

$\boxed{y^{-1} = \log_a x}$ provide $a > 0$, $a \neq 1$.

$$y = \log_a x \iff x = a^y$$

$$x = a^y$$

$$1 = \frac{dx}{dx} = \underbrace{\frac{da^y}{dy}} \cdot \frac{dy}{dx}$$

$$1 = \underbrace{a^y \cdot \ln(a)} \cdot \frac{dy}{dx} =$$

$$1 = x \ln(a) \frac{dy}{dx} =$$

$$\frac{dy}{dx} = \frac{1}{x\ln(a)}$$

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \cdot \ln(a)}}$$

Property: $\boxed{\log_a x = \dfrac{\log_e x}{\log_e a} = \dfrac{\ln x}{\ln(a)}}$ Change of Base.

Logarithmic Differentiation

$$y = f(x)^{g(x)}, \quad f(x) > 0$$

Methodology.

1) $$y = e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln f(x)}$$

$$y = e^{g(x)\ln(f(x))}$$

2) Differentiation (Chain Rule)

$$\frac{dy}{dx} = \frac{d}{dx} e^{g(x)\ln(f(x))} \quad *$$

$$= e^{g(x)\ln(f(x))} \cdot \left(g'(x)\ln(f(x)) + g(x) \cdot \frac{1}{f(x)} \cdot f'(x)\right)$$

Example: $\dfrac{dy}{dt}$, when $y = (\sin t)^{\ln(t)}$; $0 < t < \pi$.