Föreläsning 4.1_part 2

den 21 september 2020 07:53

A 4.4 - 4.5

- Critic Point
- Test with f' & f''
- Concativity up / down
- Inflection points
- Intervall halvering.

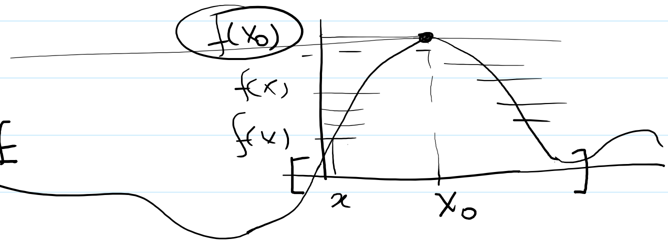
Max / Min Values

f has a max at $\underline{x_0}$ if $\underline{f(x) \leq f(x_0)}$ $\underline{\forall x \in D_f}$

the max value is called

global max value if

$f(x_0) \geq f(x)$, $\forall$ $\underline{x \in D_f}$



local max value if

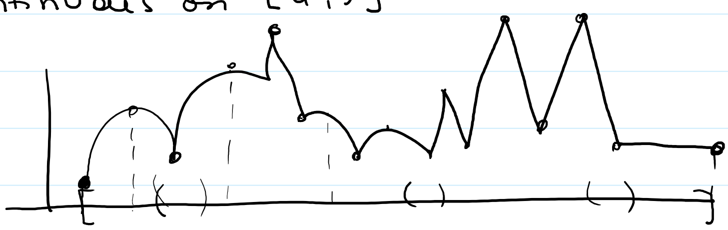
$f(x_0) \geq f(x)$ $\forall x \in I = [a, b] \subseteq D_f$.

is contained

$\underline{\text{theo}}$. (Existence of extrem values) is part of D_f

$f : [a,b] \to \mathbb{R}$ $\Rightarrow \exists$ $\underline{\text{min}}$ & $\underline{\text{max}}$ value

f continuous on $[a, b]$



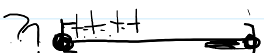
Critical Points / Singular Points / Endpoints
(1) (2) (3)

1) C.P $\Rightarrow$ $f'(x) = 0$

2) SP $\Rightarrow$ $\nexists f'(x)$, $\underline{x \in D_f}$

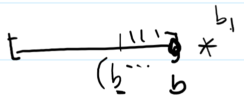
$f'(x) = +\infty$

$f'(x) = -\infty$

3) ??  $(a, a_+) \subseteq D_f$

a a_+

But $(a_-, a) \not\subseteq D_f$



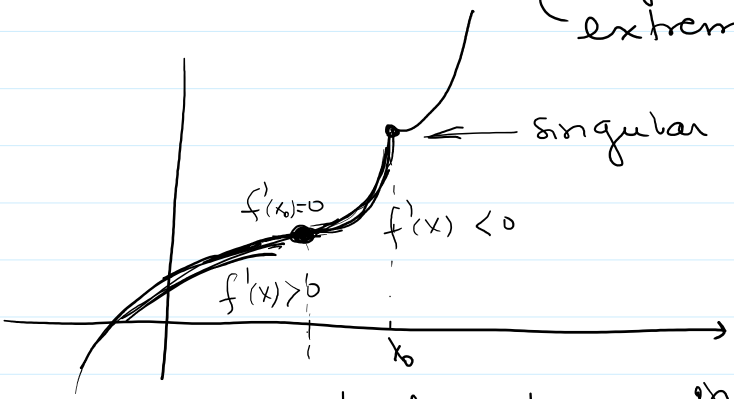
$(b_1, b_+) \not\subseteq D_f$

$(b_{-1}, b) \subseteq D_f$

Theo 6 $\quad f : [a,b] \to \mathbb{R}$

$\left[\begin{array}{l} f \text{ has a local maximun (or local min)} \\ \text{at } x_0 \in I. \end{array} \right] \Longrightarrow$

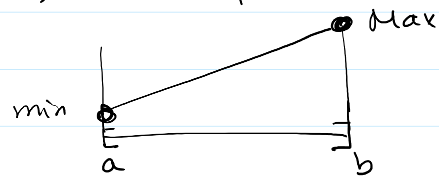
x_0 must be a $\begin{cases} \text{critical point or} \\ \text{singular point or} \\ \text{extreme point} \end{cases}$



singular point $\nexists f'(x_0)$

critical point $\longrightarrow$ inflection point.

Find Abs. Extremes
Local Extremes

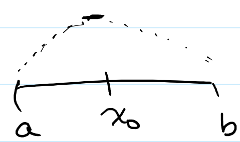


* check. the critical points $f'(x) = 0$?
singular points $\nexists f'(x)$
end points.

$\boxed{\text{Theom 7}}$ page 239 Chapter 4.

Part 1 (Testing Interior critical points & singular points)
$f : [a,b] \to \mathbb{R}$ continuous
$x_0 \in \operatorname{int}(D_f)$ interior point $\quad x_0 \in (a,b) \quad X_0 =$

i) If $x_0 \in (a,b)$ $\quad f'(x) > 0$ on (a, x_0)
$\qquad f'(x) < 0$ on (x_0, b)



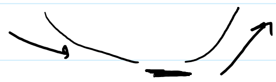
then f has a local maximum on x_0.

$f'(x) < 0$ on $(\ldots)$

Then f has a local maximum on x_0.

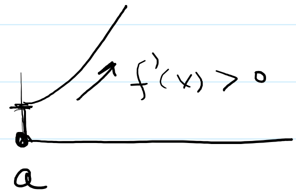
(ii) If $x_0 \in (a,b)$ $\quad f'(x) < 0$ on (a, x_0)
$\qquad\qquad\qquad\qquad f'(x) > 0$ on (x_0, b)

then f has a local minimum on x_0.



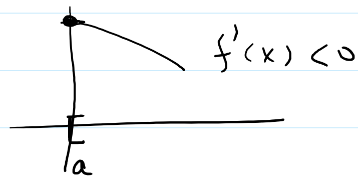
Part II Testing end Points of the Domain.

(iii) If $\left[\begin{array}{ll} f'(x) > 0 & \text{on } (a,b) \\ f'_+(a) = 0 & \end{array} \right]$



Then f has a local minimum on a.

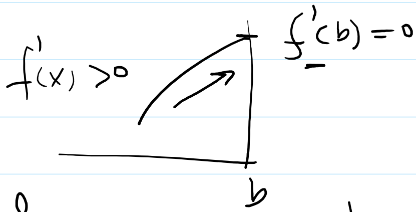
(iv) If $\left[\begin{array}{ll} f'(x) < 0 & \text{on } (a,b) \\ f'_+(a) = 0 & \end{array} \right.$



Then f has a local maximum on a.

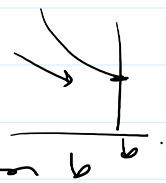
Suppose $b \equiv$ right end point

v) $f'(x) > 0$ on (a,b)



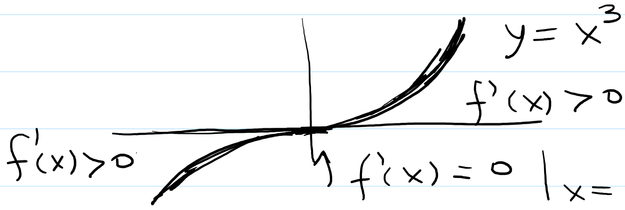
Then f has a local max on b

vi) $f'(x) < 0$ on (a,b) $\Rightarrow$



then f has a local min on b

Remark



$f'(x) = 0 \,\big|_{x=0}$ just an inflection point.

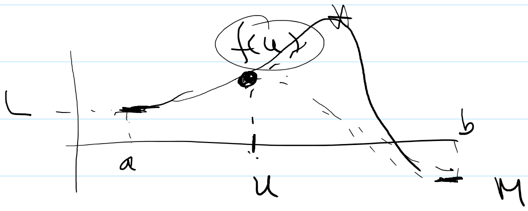
Theorem 8 page 240 (A)

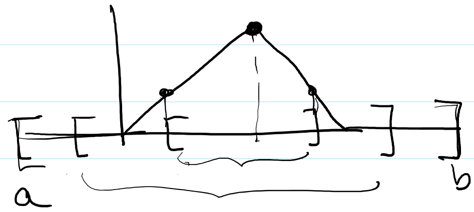
Existence of extreme values on Open Intervals

$$\begin{cases} f : (a,b) \longrightarrow \mathbb{R} \\ \lim_{x \to a_+} f(x) = L \qquad \lim_{x \to b_-} f(x) = M \end{cases}$$

i) if $f(u) > L$ & $f(u) > M$ for some $u \in (a,b)$

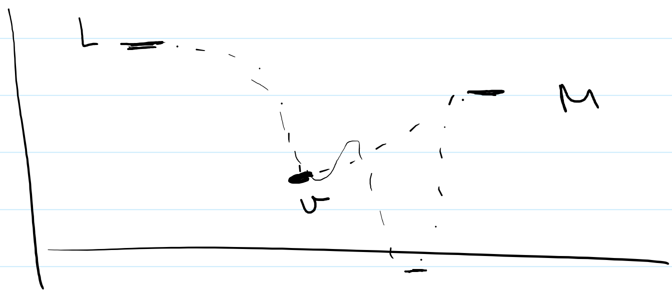
then f has an absolute Max value on (a,b)





ii) if $f(v) < L$ & $f(v) < M$, $v \in (a,b)$

then f has an absolute min-value on (a,b)



Obs: $a = -\infty$

$$\lim_{x \to a_+} f(x) = \lim_{x \to -\infty} f(x)$$

$b = +\infty$

$$\lim_{x \to b_-} f(x) = \lim_{x \to +\infty} f(x)$$

L, M can be $+\infty$ or $-\infty$

Example 5 page 241

1) Show that $f(x) = x + \frac{4}{x}$ has an absolute min value

on $I = (0, +\infty)$ (open interval)

×

on $I = (0, +\infty)$ (open interval)

2) Find that minimum.

Resolution : We consider theorem 8.

1) find $\lim_{x\to 0} f(x)$, $\lim_{x\to\infty} f(x)$

$$\lim_{x\to 0} x + \frac{4}{x} = \left[0 + \frac{4}{0}\right] = \left[\frac{4}{0}\right] = ???$$

So

$$\lim_{x\to 0_+} x + \frac{4}{x} = \underbrace{\lim_{x\to 0_+} x}_{0} + \lim_{x\to 0_+} \frac{4}{x} =$$

$$0 + 4 \lim_{x\to 0_+} \frac{1}{x} = 4\left[\frac{1}{0_+}\right] = 4 * (+\infty) = +\infty$$

$$\cancel{\lim_{x\to 0_-} x + \frac{4}{x}} = 0 + 4 \lim_{x\to 0_-} \frac{1}{x} = \left[\frac{1}{0_-}\right] = -\infty$$

Men $\xrightarrow[x\to 0_-]{\text{det är inte}}$ 0 på D_f

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x$ They have same behaviour when $x\to\infty$ $\quad D_f = (0, +\infty)$

$$\lim_{x\to+\infty} x + \frac{4}{x} \overset{*}{=} \lim_{x\to\infty} x + 4 \lim_{x\to+\infty} \frac{1}{x} \overset{*}{=} [+\infty] + 4[0_+] = +\infty$$

It will be important to draw the graph.

$f(1) = 1 + \frac{4}{1} = 5$. So, Theorem 8 garantees that f must have an absolute MIN value at some point $x_0 \in (0, +\infty)$

II. To find the min value :

1) Determine critical points / singular points (we already know that the min value is not on the end points!)

2) Evaluate f on these critical/singular points.

$$f(x) = x + \frac{4}{x}$$

$$f'(x) = 1 + 4\left(-1(x)^{-2}\right) = 1 - \frac{4}{x^2}$$

x

$f'(x) = 0 \iff 1 - \dfrac{4}{x^2} = 0 \iff \dfrac{x^2-4}{x^2} = 0, \quad x \neq 0 \quad **$

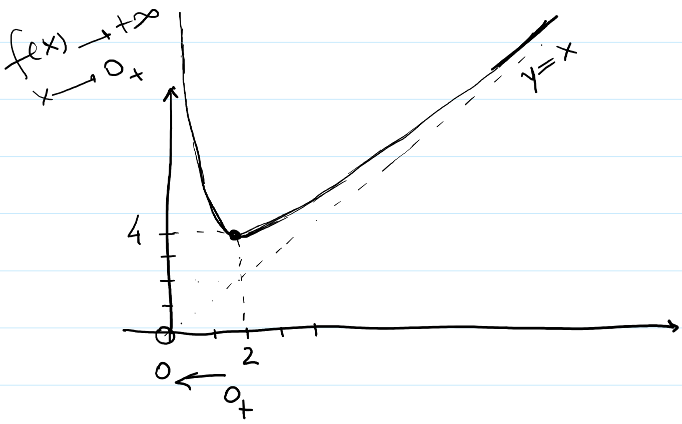
$(x^2-4) = 0 \iff (x-2)(x+2) = 0 \quad \therefore \boxed{x=2} \text{ or } \cancel{\boxed{x=-2}} \notin D_f = (0,+\infty)$

$\cancel{f'(0)} \quad x \neq 0. \quad **$ $\qquad f(x) = x + \dfrac{4}{x}$

$\cancel{f'(-2)} \quad -2 \notin (0,+\infty)$

$f(2) = (2) + \dfrac{4}{(2)} = 2+2=4$

So the min value is $f(2) = \underline{3}$ and the point $x_0 = 2$ in the domain where the min value is realized.



$x \in (0,+\infty)$

Ex6 pag 241 $\quad f(x) = x\, e^{-x^2}$ $\quad$ Find & classify critical points of f.

$D_f = \mathbb{R} = (-\infty,+\infty)$

Resolution: I. Compute $f'(x)$

II. • $f'(x) = 0$ obtaing critical points

• $\nexists f'(x)$ obtaing singular points

• $\lim\limits_{x\to+\infty} f(x)$, $\lim\limits_{x\to-\infty} f(x)$, since D_f is open.

I. $f'(x) = \dfrac{d}{dx}\Big(\underbrace{x}_{h(x)}\, \underbrace{e^{-x^2}}_{g(x)}\Big)$ $\quad \dfrac{\text{product Rule}}{\text{Chain Rule}}$ $\quad h'(x).g(x) + h(x).g'(x)$

$= 1.e^{-x^2} + x\left(e^{-x^2}.(-2x)\right) =$

$f'(x) = e^{-x^2}(1-2x^2)$

•) $f'(x) = 0 \Rightarrow e^{-x^2}(1-2x^2) = 0 \; : \; e^{-x^2} \neq 0 \; \therefore \; (1-2x^2) = 0$

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Longrightarrow x = \pm\sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}}$$

So $f(x)$ has 2 critical points:

$$x_1 = -\frac{1}{\sqrt{2}} \quad \& \quad x_2 = +\frac{1}{\sqrt{2}}$$

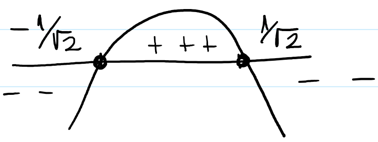
- $f(x_1) = x_1 e^{-x_1^2} = \frac{-1}{\sqrt{2}} e^{-(-1/\sqrt{2})^2} = \frac{-1}{\sqrt{2}} e^{-1/2} = -\frac{1}{\sqrt{2}\sqrt{e}} = -\frac{1}{\sqrt{2e}}$

  $f(x_2) = x_2 e^{-x_2^2} = +\frac{1}{\sqrt{2}} e^{-(1/\sqrt{2})^2} = +\frac{1}{\sqrt{2}} e^{-1/2} = +\frac{1}{\sqrt{2}\sqrt{e}} = +\frac{1}{\sqrt{2e}}$

- $f'(x) = e^{-x^2}(\underbrace{1-2x^2}) = -e^{-x^2}(x+1/\sqrt{2})(x-1/\sqrt{2})$

$f'(x) > 0 \iff -(x+1/\sqrt{2})(x-1/\sqrt{2}) > 0$

Since $e^{-x^2} = \frac{1}{e^{x^2}} > 0$ allways.



$x \in D_f$

		CP		x		CP	
x	- - -	$-1/\sqrt{2}$	- - -	0	+	$1/\sqrt{2}$	+ + + +
f'	- - - -	0	+ +	+	+	0	- - - - -
f	↘		↗				↘

x	$f(x) = xe^{-x^2}$
-3	-0,0004
-2	-0,0366
-1	-0,3679
-0.8	-0,4218
-0.6	-0,4186
-0.4	-0,3409
0	0

x	$f(x) = xe^{-x^2}$
0.4	0,3409
0.6	0,4186
0.8	0,4218
1	0,3679
2	0,0366
3	0,0004

There is a symmetry also in data.

f is an odd function because $f(-x) = -f(x)$

$$f(x) = xe^{-x^2}$$
$$f(-x) = (-x)e^{-(-x)^2} = (-x)e^{-x^2} = -(xe^{-x^2}) = -f(x)$$

$$f(x) = x e$$
$$f(-x) = (-x) e^{-(-x)^2} = (-x) e^{-x^2} = -\left(x e^{-x^2}\right) = -f(x)$$

So, this justifies the symmetry we noticed on the numerical data.

•) No singular points so far.

•) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x e^{-x^2}$ $\underline{\underline{\circledast}}$ Theorem 5 Section 3 4

Theo 5 . Section 3. 4 : The growth properties of exp and ln

If $a > 0$, then

i) $\lim\limits_{x \to +\infty} \dfrac{x^a}{e^x} = 0$

So. $\lim\limits_{x \to +\infty} x \dfrac{1}{e^{x^2}} = \lim\limits_{x \to \infty} x\left(\dfrac{1}{x} \cdot x\right) \dfrac{1}{e^{x^2}} =$

$$\lim_{x \to \infty} \left(\frac{1}{x}\right) \cdot \left(\frac{x^2}{e^{x^2}}\right) = \underbrace{\lim_{x \to \infty} \left(\frac{1}{x}\right)}_{\#} \cdot \overbrace{\lim_{x \to \infty} \left(\frac{x^2}{e^{x^2}}\right)}^{*} =$$

$$= \overset{\#}{0} \cdot \overset{*}{0} = 0$$

$$\lim_{x \to -\infty} \left(\frac{1}{x}\right)\left(\frac{x^2}{e^{x^2}}\right) = \lim_{x \to -\infty} \left(\frac{1}{x}\right) \cdot \overbrace{\lim_{x \to -\infty} \left(\frac{x^2}{e^{x^2}}\right)}^{*} =$$

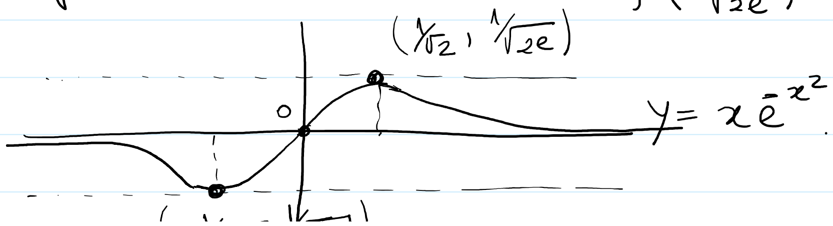
$$0_- \cdot 0_+ = 0.$$

$$* \lim_{x \to \pm\infty} \frac{x^2}{e^{x^2}} = \lim_{u \to +\infty} \frac{u}{e^u} = 0$$

Since $f(x) > 0$ at $x = 1/\sqrt{2}$,

& $f(x) < 0$ at $x = -1/\sqrt{2}$

Then f must have absolute Max & Min values by theorem 8. So $\min = f\left(-\dfrac{1}{\sqrt{2e}}\right)$ & $\text{Max} = f\left(\dfrac{1}{\sqrt{2e}}\right)$




$(-1/\sqrt{2}, -1/\sqrt{2e})$