Inversa_Trigonometriska_Funktioner (Del av F3.3)

den 23 september 2020 08:36

Remark from F.3.3

Derivative of Inverse Function pag 170 A.3.1

Suppose

$$\left[\begin{array}{l} f:(a,b)\to\mathbb{R} \text{ differentiable} \\ f'(x)>0 \;\; \forall x\in(a,b) \;\; \text{(increasing)} \\ \text{OR} \\ f'(x)<0, \forall x\in(a,b) \;\; \text{(decreasing)} \end{array}\right] \Rightarrow \left[\begin{array}{l} f \text{ is one to one on } (a,b) \\ \text{and has an inverse : } f^{-1}. \\ \text{and } \dfrac{d}{dx}(f^{-1})(x) = \dfrac{1}{f'(f^{-1}(x))} \end{array}\right]$$

Just to Remember, this result is a consequence of the Chain Rule:

$$f(f^{-1}(x)) = x$$

Differentiating both sides, we have:

$$\frac{d}{dx}\left(f(f^{-1}(x))\right) = \frac{d}{dx}x = 1 \underset{\text{Chain Rule}}{\Rightarrow} \quad f'(f^{-1}(x))\cdot\frac{d}{dx}(f^{-1}(x)) = 1 \Rightarrow$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} \qquad \square$$

So Now we are going to focus in the cases when f is a trigonometric function.

§3.5 The Inverse Trigonometric Functions

Adam pag 192

1) The 6 trigonometric functions:
   $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$
   are PERIODIC functions
   and therefore non injective.

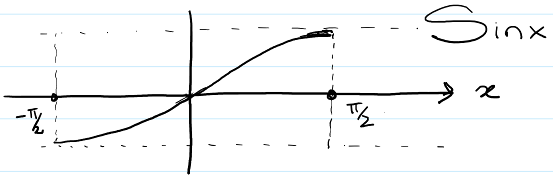
2) We can restrict their domains in such a way each one can become one-to-one and consequently invertible on that restricted interval.

3. These are the goals of this section:
   a) Obtain the domain restriction
   b) Compute the inverse function : $(f^{-1}(x))$

c) Obtain the derivative of the inverse $\frac{d}{dx}(f^{-1}(x))$ for f(x) being trigonometric.

Def 8 pag 192 : The Restricted Sine function Sinx

$$\text{Sin}x = \sin x \quad , \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



On $[-\frac{\pi}{2}, \frac{\pi}{2}]$ Sinx is one-to-one

Def 9 pag 193 The inverse of sine function : $\sin^{-1}x = \text{Arcsin}\,x$

$$y = \sin^{-1}x \iff \underline{\text{Sin}(y)} = \text{Sin}(\sin^{-1}(x)) = x \iff$$

$$x = \sin y \ , \ \text{for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Cancelation Identities:

a) $\sin^{-1}(\text{Sin}x) = \arcsin(\text{Sin}x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

b) $\text{Sin}(\sin^{-1}t) = \text{Sin}(\arcsin t) = t$ for $-1 \le t \le 1$

Observation:

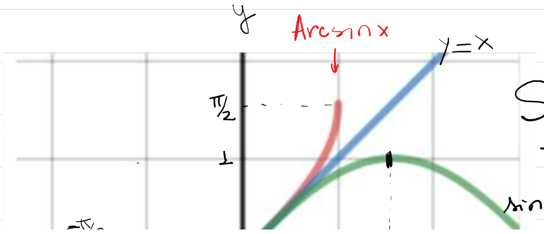
a)

$$\begin{tikzcd} x \arrow[r, "\text{Sin}(x)"] & y \arrow[r, "\arcsin(y)"] & x \\ \underbrace{-\frac{\pi}{2} \le x \le \frac{\pi}{2}}_{\text{starting point}} & -1 \le y \le 1 & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \end{tikzcd}$$

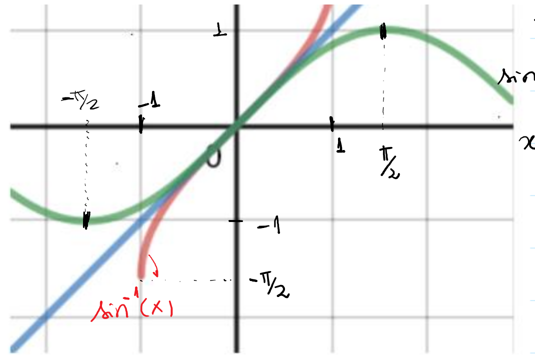
b)

$$\begin{tikzcd} t \arrow[r, "\arcsin(t)"] & x = \arcsin(t) \arrow[r, "\text{Sin}(x)"] & t \\ \underbrace{1 \le t \le 1}_{\text{starting point}} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} & -1 \le t \le 1 \end{tikzcd}$$

starting point is now no longer the Domain of Sin, but it is the Domain of $\sin^{-1}(t) = \arcsin(t)$



$$\underbrace{\underline{\text{Sin}(x) \ , \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}}}_{D_{\text{Sin}(x)}}$$



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$\underbrace{\qquad\qquad}_{D_{\sin(x)}}$

Examples: pag 193

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \pi/6$ because $\sin(\pi/6) = 1/2$ & $-\pi/2 < \pi/6 < \pi/2$.

$\pi/6 \in D_f = Im_{f^{-1}}$

$f = \sin(x)$

(b) $\sin^{-1}\left(-1/\sqrt{2}\right) = -\pi/4$ because $\sin\left(-\pi/4\right) = -\frac{1}{\sqrt{2}}$

and $-\pi/2 < -\pi/4 < \pi/2$

(c) $\sin^{-1}(2) = ?$

Not defined because $2 \notin Range_{\sin(x)} = D_{\sin^{-1}(x)}$

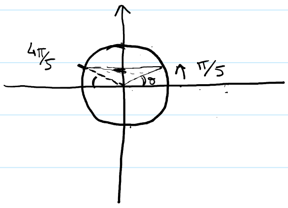
Example 2 page 193

Find (c) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$

Resolution: $\frac{4\pi}{5} \notin \left[-\pi/2, \pi/2\right]$, so we can NOT apply the cancelation identities directly.

But $\sin\frac{4\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) =$

$= \sin\left(\pi/5\right)$



Now we can consider the cancelation :

$$\sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \sin^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{5}$$

Page 194 Using Implicit Differentiation to obtain $\frac{d}{dx}\left(\sin^{-1}(x)\right)$

If $y = \sin^{-1}x$, then $\boxed{x = \sin y,}^{(A)}$ $-\pi/2 < y < \pi/2$

↓ differentiating with respect to $\underline{x}$.

$$1 = \frac{dx}{dx} = \frac{d(\sin y)}{dy} \cdot \frac{dy}{dx} \Rightarrow$$

$$1 = \cos y \cdot \frac{dy}{} \Rightarrow \frac{dy}{} = \frac{1}{}$$

Since $y \in \left(-\pi/2, \pi/2\right)$

$\cos y \neq 0$

⊗ ok!

$$1 = \cos y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} \quad \circledast \text{ ok!}$$

Since $y \in (-\pi/2, \pi/2)$

$\cos y \neq 0$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos y} \quad \circledast\circledast$$

$\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$ $\circledast\circledast$

But according to (A)

$x = \sin y \quad \therefore$

$x^2 = \sin^2 y$

Therefore

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) \overset{\circledast\circledast}{=} \frac{1}{\sqrt{1 - x^2}}$$

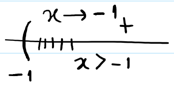
$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1 - x^2}} \qquad x \in (-1, 1)$$

Obs

$$\lim_{x \to -1_+} \frac{1}{\sqrt{1 - x^2}} \overset{\boxed{*}}{=} \frac{1}{0_+} = +\infty$$

$$\lim_{x \to 1_-} \frac{1}{\sqrt{1 - x^2}} \overset{\triangle}{=} \frac{1}{0_+} = +\infty$$

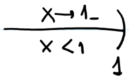
Obs 1) -1_+ $\boxed{*}$



$x > -1 \Rightarrow -x < 1$

$*\begin{cases} (x)^2 < 1^2 \quad \text{so} \\ 0 < 1^2 - x^2 \end{cases}$

Obs 2) 1_+ $\triangle$



$x < 1$

$x^2 < 1$

$0 < 1 - x^2$ ✓

Example 4 page 194.

(a) Find the derivative of $\sin^{-1}\left(\frac{x}{a}\right)$

(b) Evaluate $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx$, $a > 0$.

By the Chain Rule.

(a) $\frac{d}{dx}\left(\sin^{-1} \square\right) \overset{\circledast}{=} \frac{d}{dx}\left(f(g(x))\right) \cdot \frac{d}{dx} g'(x)$, so

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} =$$

$$= \frac{1}{\sqrt{\frac{a^2}{a^2} - \frac{x^2}{a^2}}} \cdot \frac{1}{a} = \frac{1}{\cancel{\frac{1}{a}}\sqrt{a^2 - x^2}} \cdot \cancel{\frac{1}{a}}$$

$$= \frac{1}{\sqrt{a^2 - x^2}}, \quad \text{if } \underline{\underline{a > 0}} \leftarrow \text{This is important}$$

(b) $\int f(x)\,dx = F(x) + C$ if $\frac{d}{dx}\left(F(x) + C\right) = f(x)$, $C \equiv$ real constant

So, we must find $F(x)$ such that $\frac{d}{dx}F(x) = \frac{1}{\sqrt{a^2-x^2}}$

well, but this is exactly what we have done in (a).

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \quad a>0$$

$C \in \mathbb{R}$ any constant value

Example 6 page 195.

Let $f(x) = \sin^{-1}(\sin(x))$, $\underline{x \in \mathbb{R}} \equiv D_{\sin x}$ ✓

(a) Calculate & simplify $f'(x)$

(b) where is f differentiable?

(c) where is f continuous?

(d) Sketch the graf of f.

Solution:

(a) We consider the chain Rule again since $f(x) = \sin^{-1}(g(x))$, with $g(x) = \sin x$.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\sin^{-1}(g(x))\right) \cdot \frac{d\,g(x)}{dx} =$$

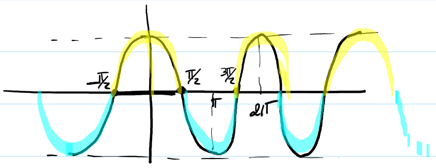
$$= \frac{1}{\sqrt{1-(g(x))^2}} \cdot g'(x) = \frac{1}{\sqrt{1-(\sin^2(x))}} \cdot \cos x =$$

$$= \frac{1}{\underset{\circledast}{\sqrt{\cos^2 x}}} \cdot \cos x = \frac{\cos x}{|\cos x|} =$$

$\circledast$ $\sqrt{a^2} = |a|$

$$= \frac{\cos x}{\cos x} = 1, \quad \text{when } \cos x > 0$$

$$= \frac{\cos x}{-\cos x} = -1, \quad \text{when } \cos x < 0$$

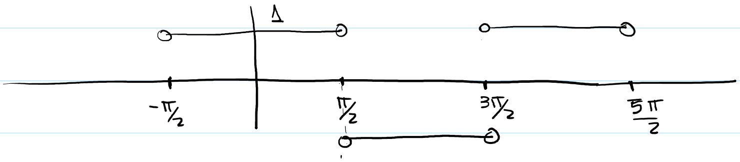


So when $\cos x = 0$, $f(x)$ is not differentiable

$\circledast$ $\cos x = 0 \Rightarrow x = -\pi/2, \pi/2, 3\pi/2, 5\pi/2 \ldots \frac{\pi}{2} + \pi . k = \frac{\pi + 2k\pi}{2}$

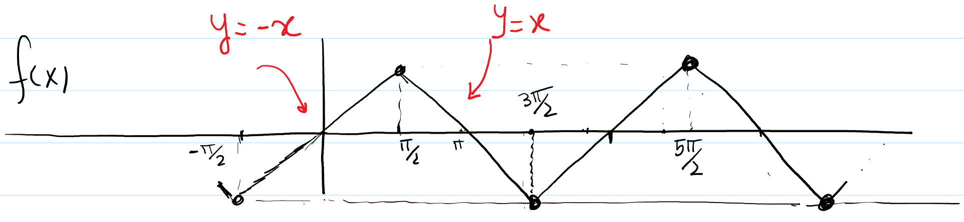
$$= \frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}.$$

$f'(x) =$



$$\text{so } f(x) = \begin{cases} x, & x \in (-\pi/2, \pi/2) \cup (\overset{3\pi/2}{-\pi/2 + 2\pi}, \overset{\frac{5}{2}\pi}{\frac{\pi}{2} + 2\pi}) \cup \ldots \end{cases}$$

so $f(x) = \begin{cases} x , & x \in (-\pi/2, \pi/2) \cup (\overbrace{-\pi/2+2\pi}, \frac{\pi}{2}+2\pi) \cup \dots \\ -x & x \in (\pi/2, 3\pi/2) \cup (\underbrace{\frac{\pi}{2} + 2\pi, \frac{3\pi}{2} + 2\pi}_{(\frac{5\pi}{2}, \frac{7\pi}{2})}) \cup \dots \end{cases}$



$\uparrow$ f is not differentiable at points $x = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$.

But f(x) is continuous, $\forall x \in \mathbb{R}$..

Now I will summarize the results from this section 3.5 here. But the main ideas are all the same and we repeat in each new case to restrict the trigonometric function to domains where the function is one-to-one and therefore in invertible.

To obtain the corresponding derivative, again we use Implicit differentiation & chain Rule.

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Def 10_ $\quad \mathrm{Tan}\,x = \tan x \;, x \in -\pi/2 < x < \pi/2 \quad \Big(x \in \overset{\text{open interval}}{\overbrace{(-\tfrac{\pi}{2}, \tfrac{\pi}{2})}} \Big)$

Def 11_ $\quad \tan^{-1} x = \mathrm{Arctan}\,x$

$$y = \tan^{-1} x \iff x = \mathrm{Tan}\,y$$
$$\iff x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Cancelation Identities

- $\tan^{-1}(\mathrm{Tan}\,x) = \arctan(\mathrm{Tan}\,x) = x$ for $x \in (-\pi/2, \pi/2)$
- $\mathrm{Tan}\,(\tan^{-1} x) = \mathrm{Tan}\,(\arctan x) = x$ for $x \in (-\infty, \infty)$

Derivative of $\tan^{-1}(x)$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

Page 197

Inverse cosine function $\cos^{-1} x$ (or $\arccos x$)

$$y = \cos^{-1} x \iff x = \cos y \qquad 0 \leq y \leq \pi.$$

obs

$$\cos y = \sin(\pi/2 - y)$$

$$y = \cos^{-1} x \iff x = \sin\left(\frac{\pi}{2} - y\right) \iff \sin^{-1} x = \frac{\pi}{2} - y =$$

$$= \frac{\pi}{2} - \cos^{-1}(x)$$

Def 12 pag 198

So: $\boxed{\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x}$ for $-1 \leq x \leq 1$.

Cancelation

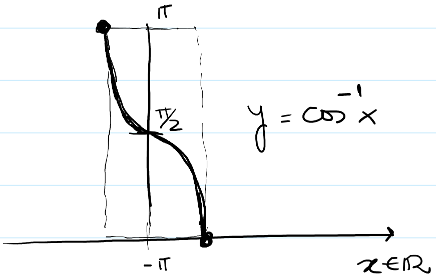
closed interval

$$\cos^{-1}(\cos x) = \arccos(x) = x \quad \text{for } x \in [0, \pi]$$

$$\cos(\cos^{-1} x) = \cos(\arccos x) = x \quad \text{for } x \in [-1, 1].$$

Derivative of $\cos^{-1}(x)$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$



Def 13 Page 198

Inverse secant function $\sec^{-1} x$ (or $\text{Arcsec}\, x$)

$$\boxed{\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) \quad \text{for } |x| \geq 1}$$

$$D_{\sec^{-1}} = (-\infty, -1] \cup [1, +\infty)$$

$$R_{\sec^{-1}} = [0, \pi/2) \cup (\pi/2, \pi]$$

Lets check out one of the cancelation identities:

$$\sec \square = \frac{1}{\cos \square} \quad ⊛$$

$$\sec(\sec^{-1}(x)) \overset{*}{=} \sec\left(\boxed{\cos^{-1}\left(\frac{1}{x}\right)}\right) \overset{⊛}{=\!=}$$

$$= \frac{1}{\cos\left(\cos^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

(what we wanted: the identity function)

for $|x| \geq 1$.

Page 199 Derivative of $\sec^{-1}(x)$

By the Chain Rule.

Page 155 Derivative of $\sec(x)$

By the Chain Rule.

$$\frac{d}{dx}(\sec^{-1}x) = \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right) = \frac{d}{dx}\cos^{-1}(g(x)) \cdot \frac{d}{dx}g'(x) =$$

$$= \frac{\cancel{-}1}{\sqrt{1-\frac{1}{x^2}}} \cdot \left(\cancel{-}\frac{1}{x^2}\right) = \frac{1}{x^2}\frac{1}{\sqrt{\frac{x^2-1}{x^2}}} = \frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} =$$

$$= \frac{1}{x^2}\frac{\sqrt{x^2}}{\sqrt{x^2-1}} = \frac{1}{x^{\cancel{2}}}\frac{\cancel{|x|}}{\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

So $$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

Definition 14 pag 199: Inverse Cosecant & Inverse Cotangent functions

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \quad (|x| \geq 1)$$

$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) \quad (x \neq 0$$