Föreläsning 4.2
_del_a
den 24 september 2020   08:00

Mål:
- → Konvexitet / konkavitet (still from F. 4.1)
- → Inflektionspunkter (still from F. 4.1)
- → Intervallhalvering (På Matlab: Bissecção Method)

- Grafritning, asymptoter
- jämförelse av mellan $e^{f(x)}$, $f(x)^{g(x)}$, $\log(h(x))$ då variabeln går mot ∞

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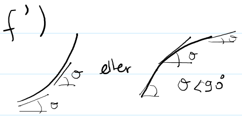
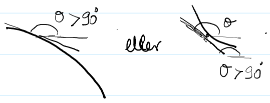
Def 3 ⊛ f is concave up on $(a,b)=I$

IF [f is differentiable on $(a,b)=I$ $(\exists f')$
&
$f'(x)>0$, $\forall x \in I$.]



⊛ f is concave down on $(a,b)=I$

IF [f is differentiable on (a,b)
&
$f'(x)<0$ $\forall x\in I$.]



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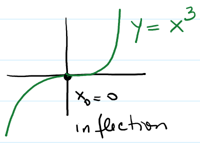
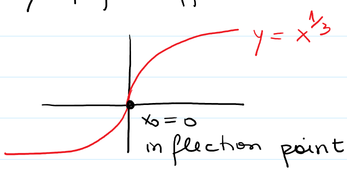
Def 4 Inflection Points

$P=(x_0, f(x_0))$ is an inflection point of the curve $y=f(x)$ (or the function f has an inflection point at x_0).

If [a) the graph of $y=f(x)$ has a tangent line at $x=x_0$
&
b) the concavity of f is opposite on opposite sides of x_0]

Ex.:





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Theorem 9 Concavity & Second Derivative

a) If [$f''(x)>0$ on I] Then [f is concave up on I].

b) If [$f''(x)<0$ on I] then [f is concave Down on I]

c) If [f has an inflection point at x_0 & $f''(x_0)$ exists] Then [$f''(x_0)=0$]

Proof:

(a), (b) follow from applying theo 12 / Section 2.8 to the derivative of $f \equiv f'$.

(c) If f has an inflection point at $x_0 \in (a,b)$ and $f''(x_0)$ exists, then f must be differentiable in an open interval containing x_0

$x_0 \in (x_0-\delta, x_0+\delta) \subset (a,b)$ $\delta>0$ any small distance.

By the Definitions Def 3 & Def 4,

the sign of $f'(x)$ on both sides of x_0

must be different. (So $f'(x)$ is increasing in one side and decreasing on the other)
So f' must have a max or a min at x_0.
Therefore by theorem 6 $(f')'(x_0) = 0$ (f' has a critical point at x_0)

Theorem 10 The second derivative Test page 245.

(a) $[f'(x_0) = 0 \ \& \ f''(x_0) < 0] \Rightarrow$ [f has a local maximum value at x_0]

(b) $[f'(x_0) = 0 \ \& \ f''(x_0) > 0] \Rightarrow$ [f has a local minimum value at x_0]

(c) $[f'(x_0) = 0 \ \& \ f''(x_0) = 0] \Rightarrow$?? No conclusions can be drawn.
f may have a local Max value at x_0
or
f may hava a local Min value at x_0
OR
f may have an inflection point instead
The investigation goes further to $f'''(x)$ ...

Page 247 §4.6 Sketching the Graph of a function.

1) Asymptotes

Def 5 pag 248 The graph of $y = f(x)$ has a vertical asymptote at $x = a$
If either $\lim_{x \to a_-} f(x) = +\infty$ or $-\infty$ OR $\lim_{x \to a_+} f(x) = +\infty$ or $-\infty$ OR BOTH.

Example 1. $f(x) = \dfrac{1}{\boxed{x^2 - x}}$ Does $f(x)$ have asymptote ?

1) * If denominator has roots, these are values we have to investigate

$x^2 - x = 0 \Rightarrow x(x-1) = 0 <$ $x_1 = 0$, $x_2 = +1$

$\boxed{x_1 = 0}$ a) $\lim_{x \to 0_-} \dfrac{1}{x^2 - x} = \dfrac{1}{0_+ - (-0)} = \dfrac{1}{0_+ + 0_+} = \dfrac{1}{0_+} = +\infty$

$\underset{0_-}{\longrightarrow} 0$ $\boxed{x < 0} \Rightarrow -x > 0$, $(x^2) > 0$

yes! $f(x)$ has an asymptote at $x_1 = 0$

b) $\lim_{x \to 0_+} \dfrac{1}{x^2 - x} = \dfrac{1}{0_-} = -\infty$ Important to justify.

0_+ $x > 0$ $0 < x < 1 \Rightarrow 0 < x^2 < x \Rightarrow x^2 - x < 0$

$\boxed{x_2 = 1}$ a) $\lim_{x \to 1_-} \dfrac{1}{x^2 - x} = \dfrac{1}{0_-} = -\infty$

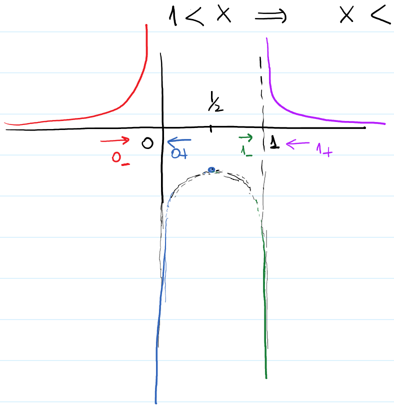
$\boxed{x_2=1}$ **a)** $\lim\limits_{x\to 1_-} \dfrac{1}{x^2-x} = \dfrac{1}{0_-} = \boxed{-\infty}$

$x_- \to 1$ $\quad 0<x<1 \Rightarrow x^2<x \Rightarrow \underline{x^2-x<0}$

b) $\lim\limits_{x\to 1_+} \dfrac{1}{x^2-x} = \dfrac{1}{0_+} = \boxed{+\infty}$

$\xleftarrow{x_+}$ 1 $\quad x>1$

$1<x \Rightarrow x<x^2 \Rightarrow 0<x^2-x$



$f(\tfrac{1}{2}) = \dfrac{1}{(0.5)^2-0.5} = \dfrac{1}{0.25-0.5} = \dfrac{1}{-0.25} = -4$

Def 6. page 249 — The graph of $y=f(x)$ has a horizontal asymptote $y=L$ If

either $\lim\limits_{x\to+\infty} f(x) = L$ OR $\lim\limits_{x\to-\infty} f(x) = L$ or Both

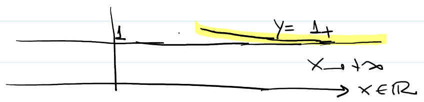
Example Find asymptotes of $y=f(x)$

$$f(x) = \frac{x^4+x^2}{x^4+1}$$

$$\lim_{x\to+\infty} \frac{x^4+x^2}{x^4+1} = \lim_{x\to+\infty} \frac{\cancel{x^4}(1+1/x^2)}{\cancel{x^4}(1+1/x^4)} = \frac{(1+0_+)}{(1+0_+)} = 1_+$$

So $y=1$ is a horizontal asymptote

$1 > \dfrac{1}{x^2} \overset{?}{>} \dfrac{1}{x^4}$

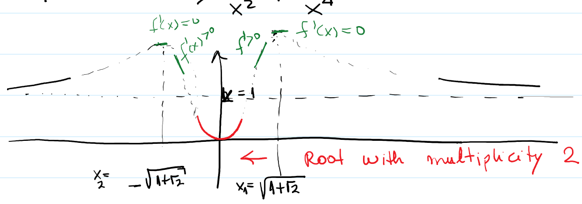


yes! Because $x^4 > x^2 > 1$

$$\lim_{x\to-\infty} \frac{x^4+x^2}{x^4+1} = \lim_{x\to-\infty} \frac{\cancel{x^4}(1+1/x^2)}{\cancel{x^4}(1+1/x^4)} = \frac{1+0_+}{1+0_+} \quad \text{again}$$

$\cancel{1}+\dfrac{1}{x^2} > \cancel{1}+\dfrac{1}{x^4}$?

yes! $\quad 1 > \dfrac{1}{x^2} > \dfrac{1}{x^4} \Leftrightarrow 1<x^2<x^4$ ✓



← Root with multiplicity 2 $\Rightarrow$ f must be as x^2 next to $x=0$.

$f(x) = \dfrac{x^4+x^2}{x^4+1}$ $\quad x^4+1=0 \quad x^4=-1$ $\therefore$ So, $\nexists\, x\in\mathbb{R} \,/\, x^4=-1$.

denominator allways positive.

$f(x)=0 \Rightarrow x^4+x^2=0 \quad x^2(x^2+1)=0$

$x = 0$ root med multiplicity $= 2$.

$x^2+1=0 \Rightarrow x^2 = -1$??

$\nexists\, x \in \mathbb{R},\ x^2=-1$

So now we need to find out what the function does in a vicinity of zero. $(0-K, 0+K)$ for some $K>0$ constant.

Let's verify if f has critical points

c) $f'(x)=0$ $\qquad f(x) = \dfrac{x^4+x^2}{x^4+1} = \underbrace{(x^4+x^2)}_{f(x)}\underbrace{((x^4+1)^{-1})}_{g(x)}$

$$f'(x) = (4x^3+2x)\frac{1}{(x^4+1)} + (x^4+x^2)(-1)(x^4+1)^{-2}\cdot(4x^3)$$

$$f'(x) = \frac{4x^3+2x}{x^4+1} - \frac{(x^4+x^2)}{(x^4+1)^2}\cdot 4x^3$$

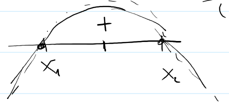
$$f'(x) = \frac{(4x^3+2x)(x^4+1) - 4x^3(x^4+x^2)}{(x^4+1)^2} =$$

$$f'(x) = \left[\left(\cancel{4x^3 x^4} + 4x^3\cdot 1 + 2x\cdot x^4 + 2x\right) - \left(\cancel{4x^3 x^4} + 4x^3\cdot x^2\right)\right]\frac{1}{(x^4+1)^2}$$

$$f'(x) = \frac{\left[2x^5 + 4x^3 + 2x - 4x^5\right]}{(x^4+1)^2} = \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2} =$$

$$f'(x) = -2x\frac{(x^4-2x^2-1)}{(x^4+1)^2}$$

$f'(x) = \dfrac{-2x(x^4-2x^2-1)}{(\ )^2}$



$f'(x)=0 \Rightarrow -2x=0$ or $x=0$

$x^4-2x^2-1=0$

$z = x^2 \Rightarrow z^2-2z-1=0$

$$z_{1,2} = \frac{2\pm\sqrt{4-4.1.(-1)}}{2} < \begin{matrix} \dfrac{2+\sqrt{8}}{2} = \dfrac{2+2\sqrt{2}}{2} \\[2ex] \dfrac{2-\sqrt{8}}{2} = \dfrac{2-2\sqrt{2}}{2}\end{matrix}$$

$z_1 = 1+\sqrt{2} > 0$

$z_2 = 1-\sqrt{2} < 0$

$$\boxed{x^2 = z_1 \Rightarrow x^2 = 1+\sqrt{2} < \begin{matrix} x_1 = \sqrt{1+\sqrt{2}} \\ x_2 = -\sqrt{1+\sqrt{2}}\end{matrix}}$$

$f'(x_1)=0$ & $f'(x_2)=0$ which are also maximum values of $f(x)$.

$\square$

Page 250 Asymptotes of a Rational function

$f(x) = \dfrac{P_m(x)}{Q_n(x)}$; $\quad P_m(x)$ = polynomial of degree m

$Q_n(x)$ = polynomial of degree n.

Assume that P_m & Q_n have

No common linear factors

(A) If $Q_n(x_0)=0$ Then the graph of f has a Vertical asymptote at x_0.

(B) If $m<n$, Then the graph of f has a two-sided Horizontal asymptote $y=0$.

(C) If $m=n$, Then the graph of f has a two-sided Horizontal asymptote $y=L$ $(L\neq 0)$ $L=$ quotient of the coefficients associated with m & n.

(D) If $m=n+1$, then the graph of f has a two-sided oblique asymptote. This can be found

$$f(x)=\frac{P_m(x)}{Q_n(x)} = \underline{ax+b} + \frac{R(x)}{Q_n(x)}$$

the oblique asymptote is $y=\underline{ax+b}$

(E) If $m>n+1$, then the graph of f has No horizontal NOR oblique asymptotes

<u>Page 251</u> <u>Check list</u> for curve sketching

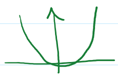
1) Obtain $f'(x)$ & $f''(x)$ written in factored form.

2) Examine $f(x)$: obtain D_f, V_f

2.a) Look at the zeros of denominators $\Rightarrow$ Vertical asymptotes.

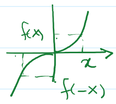
2.b) $\lim\limits_{x\to\pm\infty} f(x) \Rightarrow$ horizontal or oblique asymptotes

2.c) $f(x)=f(-x)$ f is even



symmetry with respect to $\overline{oy}$.

$f(x)=-f(-x)$ f is odd



antisymmetry with respect to $\overline{ox}$.

2d) Obvious points $f(x)\cap\overline{ox}$ (Roots)

$f(x)\cap\overline{oy}$ $(f(0))$

3. $f'(x)$, $D_{f'}$. $V_{f'}$

3.a) $f'(x)=0$, $\nexists f'(x)$, extremes of the interval (Critical, singular or endpoints)

3b) $f'(x)>0 \Rightarrow f$ is increasing

$f'(x)<0 \Rightarrow f$ is decreasing

4. $f''(x)$

4.a $f''(x)=0$

4.b $f''(x)$ undefined (This will include singular points
end points
vertical asymptotes

4.c $f''(x) > 0$
$f''(x) < 0$ Concativity

4.d Any inflection points.