

§3.4

Exponential & Logarithmical Growth

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$$\underbrace{e^x}_{\text{poly}} \times \underbrace{\text{polyn}}_{\text{poly}} = a_n x^n + \dots$$

$$\underbrace{\text{poly}}_{\text{poly}} \times \ln x =$$

pag 185 Theorem 4

If $x > 0$, then $\ln x \leq x - 1$

Proof

So, $g'(x) > 0$ et $g(x) = \ln(x) - (x-1)$ for $x > 0$.

$$g(1) = \underbrace{\ln(1)}_0 - (1-1) = 0 \quad g(1) = 0$$

$$g'(x) < 0$$

$$g'(x) = \frac{1}{x} - 1 \quad \left\{ \begin{array}{l} \frac{1}{x} > 1 \Rightarrow \frac{1}{x} > 1 \Rightarrow x < 1 \\ \frac{1}{x} < 1 \Rightarrow \frac{1}{x} < 1 \Rightarrow x > 1 \end{array} \right.$$

increasing on $(0,1)$

$$g(x) < g(1) \quad , \quad \forall x \in (0,1) \Rightarrow \boxed{1 > x > 0}$$

$$\ln(x) - (x-1) < 0 \Rightarrow \ln(x) < (x-1)$$

$$g(1) = 0 \Rightarrow$$

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Theorem 5

$$0 \quad \forall x \in (0,1) \Rightarrow g(x) \text{ is}$$

$$\frac{1}{x} < 1$$

$$\frac{1}{1} < x \quad \boxed{x > 1}$$

If $a > 0$

$$(a) \lim_{x \rightarrow +\infty}$$

$$, \quad \forall x \in (1, +\infty) \Rightarrow g(x) \text{ is decreasing on } (1, +\infty)$$

$$g(1) < g(x) \quad \forall x \in (1, +\infty)$$

$$0 < \ln(x) - (x-1) \Rightarrow$$

$$(x-1) < \ln(x)$$

$$\ln(x) = x-1 \quad |_{x=1}$$

$$\text{so } \ln(x) \leq (x-1) \quad \forall x > 0$$

$$(c) \lim_{x \rightarrow -\infty}$$

$$(d) \lim_{x \rightarrow 0^+}$$

The growth properties of Exp & ln.

then

$$\frac{x^a}{e^x} = 0 \quad \left(\lim_{x \rightarrow \infty} \text{polynomial faster than } e^x \right)$$

Proof

(1) we

Tryck:

$$\frac{\ln x}{x^a} = 0 \quad \left(\text{polynomial} \rightarrow \infty \text{ much faster than } \ln x \right)$$

Since

using $|x|^a e^x = 0$

// In a struggle between x^a & e^x , e^x wins!

0 $x^a \ln x = 0$

New

between x^a & $\ln$, x^a wins!

start proving (b) $\lim_{x \rightarrow \infty} \frac{\ln x}{x^a} = 0$

0 Let $x > 1$, $a > 0$ $S = \frac{a}{2}$ ($a = 2S$)
 $\ln(x^S) = S \ln x$, we have

theo 4 that
 $0 < S \ln x = \ln(x^S) \leq x^S - 1 < x^S$
 $\frac{0}{S} < \frac{S \ln x}{S} < \frac{x^S}{S} \Rightarrow 0 < \ln x < \frac{x^S}{S}$

By the

$0 < \ln x < x^S \cdot \frac{1}{S} \Rightarrow 0 < \frac{\ln x}{x^a} < \frac{x^S}{S x^{2S}} = \frac{1}{S x^S}$

Proof (b) —

$0 < \frac{\ln x}{x^a} < \frac{1}{S x^S}$
 $\downarrow x \rightarrow \infty$

Squeeze Theorem

$\lim_{x \rightarrow \infty} \frac{\ln x}{x^a} = 0$

→ (d) Goal: $\lim_{x \rightarrow 0_+} x^a \ln x = 0$

using that $\lim_{x \rightarrow \infty} \frac{\ln x}{x^a} = 0$

Hypothesis

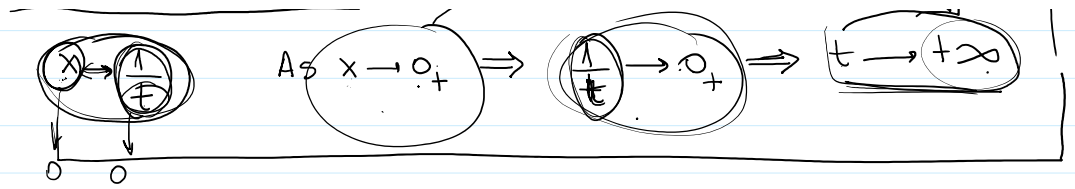
⊗ Change of variable

$x \rightarrow 0_+$

As $x \rightarrow 0_+$

$\frac{1}{t} \rightarrow 0_+$

$t \rightarrow +\infty$



So $\lim_{x \rightarrow 0_+} x^a \ln x \stackrel{\text{L'Hôpital}}{=} \lim_{t \rightarrow \infty} \left(\frac{1}{t}\right)^a \ln\left(\frac{1}{t}\right) =$

$\lim_{t \rightarrow \infty} \frac{\frac{1}{t^a} \ln(t)}{\ln(t)} \stackrel{\text{L'Hôpital}}{=} \lim_{t \rightarrow \infty} \frac{\frac{1}{t^{a+1}}}{\frac{1}{t}} = \lim_{t \rightarrow \infty} \frac{1}{t^a} = 0$

done before (by (b))