Mål: → Conclude § 3.4
→ A. 4.9 Linear Approximations
→ A. 4.10 Taylor polynom.

§ 3.4 Theo. 5 page 185

If $a>0$, then

a) $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$

b) $\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$ ← Proof done in $\underline{F. 4.2}$ (using Squeeze Theorem)

c) $\lim_{x\to-\infty} |x|^a \cdot e^x = 0$

d) $\lim_{x\to 0_+} x^a \ln x = 0$ ← Proof done based on (b) & Change of variable

Proof $\underbrace{(b)}_{\text{Hypothesis}} \Rightarrow \underbrace{(a)}_{\text{thesis}}$

If $x = \ln t$ then $t\to\infty$ as $x\to\infty$. So

$$\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{t\to\infty} \frac{(\ln t)^a}{e^{\ln t}} = \lim_{t\to\infty} \frac{(\ln t)^a}{t} = \quad * \; t = t^1 = t^{a/a} = (t^{1/a})^a$$

$$= \lim_{t\to\infty} \left(\frac{\ln t}{t^{1/a}}\right)^a \underset{\text{By continuity}}{=} \left(\lim_{t\to\infty} \frac{\ln(t)}{t^{1/a}}\right)^a \underset{\text{By (b)}}{=} (0)^a = (0). \quad \square$$

Proof (a) ⇒ (c)

(a) $\lim_{x\to\infty} \frac{x^a}{e^x} = 0$, $a>0$.

Consider the following Change of variable: $x = -t$

So $\lim_{x\to-\infty} |x|^a e^x = \lim_{t\to+\infty} |-t|^a \cdot e^{-t} =$

$\lim_{t\to\infty} \frac{|-t|^a}{e^t}$ since $|-t| = t$, $t\to+\infty$, we have:

$\lim_{t\to\infty} \frac{t^a}{e^t} = 0$ (by) (b).

Theorem 6 page 189

$$\boxed{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$$

Proof:

Case 1: if $x=0$ $\lim_{n\to\infty}(1+0)^n = 1 = e^0 = 1.$

Case 2 $x\neq 0$. Let's assume the change of variable $h=\frac{x}{n}$ so $n\to\infty \Rightarrow h\to 0.$

In this way, we have

$$\lim_{n\to\infty} \ln\left(\left(1+\frac{x}{n}\right)^n\right) = \lim_{h\to\infty} n \ln\left(1+\frac{x}{n}\right) =$$

$$= \lim_{n\to\infty} n\cdot\frac{x}{x}\ln\left(1+\frac{x}{n}\right) = x\lim_{n\to\infty} \frac{n\ln\left(1+\frac{x}{n}\right)}{x}$$

$$= x\cdot\lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\left(\frac{x}{n}\right)}$$ where $h=\frac{x}{n}$ $=$

$n\to\infty \Rightarrow h\to 0$

$$= x\lim_{h\to 0}\left(\frac{\ln(1+h)}{h}\right).$$

Since $\ln(1)=0$, we have:

$$= x\lim_{h\to 0}\left(\frac{\ln(1+h)-\ln(1)}{h}\right)$$

$$= x\left(\frac{d}{dt}\ln(t)\right)\Big|_{t=1}$$ By the definition of derivative

$$= x\,\frac{1}{t}\Big|_{t=1} = x\cdot 1 = x..$$

(*)

$$= x \left. \frac{1}{t} \right|_{t=1} = x \cdot 1 \underset{\circledast}{=} x.$$

Since the $\ln x$ function is differentiable, it is continuous.
Hence, by theorem 7 of section 1.4

$$\ln\left(\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \right) = \lim_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n \stackrel{\circledast}{=} x$$

But this implies that

$$e^{\ln\left(\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \right)} = e^{x}$$

$$\shortparallel$$

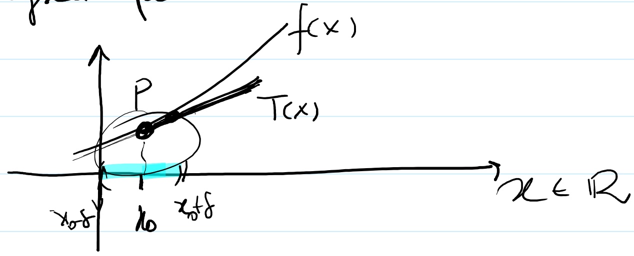
$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$$

$\square$

§ 4.9 Linear Approximations.

page 269

One powerfull application of the derivative is its capability of providing a local linear approximation for the original function, obtained by the tangent line in the vicinity of the analysed point.



Definition 8
page 270

The linearization of the function f about a point a is the function L defined by (the tangent line) page 270

is the function L defined by (the tangent line)
$L(x) = f(a) + f'(a)(x-a)$

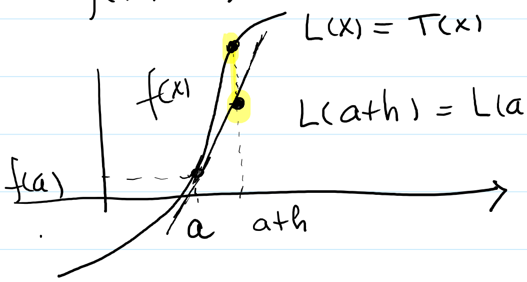
<u>Obs</u> We say that $f(x) \approx L(x) = f(a) + f'(a)(x-a)$ provides linear approximations for values of f <u>near a</u>.

<u>Obs</u> When we discussed about differentials

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \iff \Delta y \approx \frac{dy}{dx} \cdot \Delta x.$$

And $\Delta y = f(a+\Delta x) - f(a)$ small changes for f where associated with small changes on the domain.

$$f(x+a\Delta) \simeq L(a+\Delta x) = f(a) + f'(a)\Delta x$$



$L(a+h) \approx f(a+h)$

Error Analysis

$$\boxed{\text{Error} := \text{true value} - \text{approximate value}.}$$

$f(x) = L(x) = f(a) + f'(a)(x-a)$

Theorem 11 En Error formula for linearization
page 272

<u>If</u> $f''(t)$ exists for all $t \in (a-\delta, a+\delta)$, $x \in (a-\delta, a+\delta)$

<u>Then</u> $\exists\, s$, $a-\delta < a < s < x < a+\delta$, such that the Error $E(x) = f(x) - L(x)$ in the lineara

approximation $f \approx L(x) = L(a) + f'(a)(x-a)$ satisfies

$$E(x) = \frac{f''(s)}{2!}(x-a)^2$$

<u>Proof</u>: <u>Assumption</u>: $x > a$. (in case $x < a$, the proof is analogous)

$$E(t) = f(t) - L(t)$$
$$= f(t) - (f(a) + f'(a)(t-a))$$

$E(a) = f(a) - f(a) - 0 = 0$

$$E'(t) = f'(t) - f'(a)$$

By the Generalized Mean Value Theorem (The 16, A.2.8) applied to $E(t)$ and $(t-a)^2$ on $[a,x]$, we can obtain:

By the G.MVT.

$$(A) \quad \frac{E(x)}{(x-a)^2} = \frac{E(x) - \overbrace{E(a)}^{=0}}{(x-a)^2 - \underbrace{(a-a)^2}_{=0}} = \frac{E'(u)}{2(u-a)} = \frac{f'(u) - f'(a)}{2(u-a)} = \frac{1}{2} f''(s),$$

for some $s \in (a,u)$.

$$(A) \Rightarrow \frac{E(x)}{(x-a)^2} = \frac{1}{2} f''(s) \quad \therefore \quad \boxed{E(x) = \frac{1}{2} f''(s)(x-a)^2}$$

□

<u>Corollary A</u> If $f''(t)$ has constant singn $\forall t \in (a,x)$ then $E(x)$ has the same sign.

$$f''(t) > 0 \Rightarrow f(x) > L(x)$$
$$f''(t) < 0 \Rightarrow f(x) < L(x)$$

Corollary B If $|f''(t)| < K$ for $t \in (a,x)$ $K \equiv$ constant
Then $|E(x)| < \frac{K}{2}(x-a)^2$

Corollary C : i) If $f''(t)$ satisfies $M < f''(t) < N$ $\forall t \in (a,x)$ $M,N \equiv$ constants
then

$$L(x) + \frac{M}{2}(x-a)^2 < f(x) < L(x) + \frac{N}{2}(x-a)^2$$

(ii) If M & N have the same sign, we can obtain a better approximation to $f(x)$ considering the mid-point of this interval containing $f(x)$.

f is approximated by

$$f(x) \approx L(x) + \left(\frac{M+N}{4}\right)(x-a)^2$$

And the error is $|\text{Error}| < \frac{N-M}{4}(x-a)^2$

Example. i) Determine the sign and estimate the size of the error in the approximation $\sqrt{26} \approx 5.1$ obtained by the linearization of $\sqrt{x}$ near $a = 25$.

ii) Use this to estimate a small interval that surelly contains $\sqrt{26}$.

Resolution.

ø) Initially, let's construct the linearization of $f(x) = \sqrt{x}$ to confirm that the approximated value 5.1 came from the linearization.

i) We compute the corollary result

ii) We estimate the interval.

ø) $f(x) = \sqrt{x}$ $\qquad a = 25 \qquad f(a) = \sqrt{25} = 5$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}} \qquad f'(a) = \frac{1}{2}\frac{1}{\sqrt{25}} = \frac{1}{25} = \frac{1}{10} = \underline{0{,}1}$$

$L(x) = f(25) + f'(25)(x-25)$ $\quad$ Linear approximation of $f(x) = \sqrt{x}$

$$L(26) = 5 + 0{,}1\,(26-25) = 5 + 0{,}1 = \underline{5{,}1}$$

Thus $\sqrt{26} \approx L(26) = 5.1$.

i) Now, we need to compute $f''(x)$ to in order to estimate the error

$$f(t) = \sqrt{t} = t^{1/2}$$

$$f'(t) = \frac{1}{2}\frac{1}{\sqrt{t}} = \frac{1}{2}t^{-1/2}$$

$$f''(t) = \frac{1}{2}\left(-\frac{1}{2}\right)t^{-1/2-1} = -\frac{1}{4}t^{-3/2} = -\frac{1}{4}\frac{1}{\sqrt[2]{t^3}}$$

Now

for $25 < t < 26$, we have:

a) $f''(t) < 0$ $\left(f''(25) = -\frac{1}{4}\frac{1}{\sqrt[2]{(25)^3}} \; \& \; f''(26) = -\frac{1}{4}\frac{1}{\sqrt[2]{(26)^3}}, \text{ both } < 0\right)$

So, By Corollary A

$$f''(t) < 0 \Rightarrow f(x) < L(x) \quad \therefore \underline{\sqrt{26} < 5{,}1}.$$

b) $\left|f''(25)\right| = \frac{1}{4}\frac{1}{125} = \frac{1}{500}$

$25 < 26$

$\sqrt{25} < \sqrt{26} \Rightarrow \sqrt{25^3} < \sqrt{26^3}$

$\Rightarrow |f''(x)| < \frac{1}{500} = k.$

$k = \boxed{\frac{1}{\sqrt[2]{25^3}}} > \frac{1}{\sqrt[2]{26^3}}$

Now, By Corollary B

$$|E(x)| < \frac{K}{2} \cdot (x-a)^2$$

$$|E(x)| < \frac{1}{2} \cdot \frac{1}{500}(x-a)^2$$

$$|E(26)| < \frac{1}{1000}(26-25)^2 = \frac{1}{1000} = 0{,}001.$$

1) $f(x) < L(x)$ (By $\underline{C.A}$) $\therefore \boxed{\sqrt{26} < 5.1}$

2) $|E(26)| < 0{,}001$ (By C.B)

$-0{,}001 < \sqrt{26} - 5.1 < 0{,}001 \Rightarrow$

$$-0{,}001 < \sqrt{26} - 5.1 < 0{,}001 \Rightarrow$$

$$5{,}1 - 0{,}001 < \sqrt{26} < 5{,}1 + 0{,}001 \Rightarrow$$

$$5{,}099 < \sqrt{26} < \underline{\underline{5{,}1001}}$$

But, we can keep the Best info on the right

So $\sqrt{26} \in (5{,}099,\ 5{,}1)$