Föreläsning 5.1

den 28 september 2020 08:00

Mål :
- Taylor polynom § 4.10
- Big-O
- Evaluating limits of Indeterminate form $[\frac{\infty}{\infty}]$, $[\frac{0}{0}]$
- L'Hôpital Rule $[\frac{0}{0}]$, $[\frac{\infty}{\infty}]$ § 4.2
- (*) Newton Method → Matlab.

§ 4.10 page 275 Taylor Polynomials.

1) $f(x)$ $x \in D_f = [a,b]$, $\exists f'(x)$, $\forall x \in I = (a,b)$

$$P_1(x) = L(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1$$

This is the Best linear approximation of f at $x = a$.

$$E_1(x) = f(x) - P_1(x) = \frac{f''(\xi)}{2!}(x-a)^2 \qquad \xi \in (a,x)$$

So : $\boxed{f(x) = P_1(x) + E_1(x)}$

2) $P_2(x)$ = quadratic polinomial matching $f(a), f'(a), f''(a)$.

So, $\begin{cases} P_2(a) = f(a) \\ P_2'(a) = f'(a) \\ P_2''(a) = f''(a) \end{cases}$

* Newton Method:

is a discrete version of P_2 with this matching property:

Since $f'(a) \approx \frac{f(b)-f(a)}{b-a} \approx \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{f(x+h)-f(x)}{h}$

& $f''(a) \approx \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{2h} = \frac{f(x+h) - 2f(x) + f(x-h)}{2h^2}$

$$P_2(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$$

$$E_2(x) = \frac{f'''(\xi)}{3!}(x-a)^3 \qquad \xi \in (a-\delta, x+\delta) \qquad \delta > 0$$

And this process can continue if $f', f'', \dots, f^{(n)}$ exist.

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}, \qquad \xi \in (a-\delta, x+\delta)$$

$P_n =$ n-order Taylor polynomial for f about $x=a$.

Obs: If $a=0$, the Taylor polynomial is called Maclaurin polynomial (Maclaurin approximation for f about $x=0$)

Theorem 12

page 278 If $\left[\begin{array}{l} f'(t), f''(t), \dots, f^{(n)}(t) \text{ exist } \forall t \in I = (a-\delta, x+\delta),\ \delta>0. \\ P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{array} \right]$

Then $\left[\begin{array}{l} \text{the error} \\ E_n(x) = f(x) - P_n(x) \\ \quad = \underbrace{\frac{f^{(n+1)}(s)}{(n+1)!}}_{k} \underbrace{(x-a)^{n+1}}_{u(x)} \quad s \in (a-\delta, x+\delta) \quad \delta>0 \end{array} \right.$

$$\boxed{f(x) = \underbrace{P_n(x)}_{\text{Taylor Polynomial}} + \underbrace{E_n(x)}_{\text{Lagrange Remainder}}}$$

$f(x) \approx P_n(x)$

$f(x) = P_n(x) + E_n(x)$

$\Leftrightarrow$

$E_n(x) = f(x) - P_n(x)$

Page 280

Def 9. Big-O notation.

1) we write $f(x) = O(u(x))$ as $x \to a$ if $|f(t)| \le k|u(t)|$, $k=$ constant $t \in (a-\delta, x+\delta)$

"f is a big O of u" as x approaches a

2) $f(x) = g(x) + O(u(x))$ as $x \to a$ if

$f(x) - g(x) = O(u(t))$ as $x \to a$

$|f(x) - g(x)| \le K(u(x))$, $x \to a$

Some Maclaurin Polynomials (Taylor polynomials when $a=0$) with errors given in the Big-O notation.

$K x^{n+1} = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{K} \cdot x^{n+1}$

As $x \to 0$

1) $$e^x = \underbrace{\overbrace{1}^{f(0)} + \overbrace{x}^{\frac{f'(0)}{1!}} + \overbrace{\frac{x^2}{2!}}^{f''(0)} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}}_{P_n(x)} + \underbrace{O(x^{n+1})}_{E_n(x)}$$

$f^{(k)}(x) = e^x, \forall k$

$f^{(k)}(0) = 1, \forall k$

$$\underbrace{\qquad 2! \qquad 3! \qquad \qquad n!}_{P_n(x)} \qquad \underbrace{\qquad}_{E_n(x)} \qquad f^{(k)}(0) = 1, \forall k$$

2) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots + (-1)^n \dfrac{x^{2n}}{(2n)!} + O(x^{2n+2})$

3) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots + (-1) \dfrac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3})$

4) $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + O(x^{n+1})$

5) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dots + (-1)^n \dfrac{x^n}{n} + O(x^{n+1}) \leftarrow$

6) $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + O(x^{2n+3})$

Page 280 theorem 13 : The Taylor Polynomial is unique !

Exercises : E(21) (Övningsföreläsning på fredag)

? $f(x) = \sin x$ i) $\sin(3,14) = ?$ $a = \pi$ Error? $I = ?$

i) $f(x) \simeq \underbrace{L(x)}_{P_1(x)} = f(a) + \dfrac{f'(a)}{1!}(x-a)$

$f(x) = \sin x \longrightarrow f(\pi) = \sin(\pi) = 0$

$f'(x) = \cos x \qquad f'(\pi) = \cos(\pi) = -1$

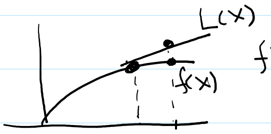
$f''(x) = -\sin x \qquad \boxed{f''(\pi) = -\sin(\pi) = 0}$

$f'''(x) = -\cos x \qquad f'''(\pi) = -\cos(\pi) = -(-1) = +1$

i) $f(x) \simeq L(x) = 0 + (-1)(x - \pi) = \pi - x$

$f(3,14) \cong L(3,14) = \pi - 3,14 = 0,0015927$ ✓

(ii) Error ?



$f'' < 0 \Rightarrow f(x) < L(x)$

Error : $f(x) - L(x) = \dfrac{f''(\xi)}{}(x-a)^2$

Error : $f(x) - L(x) = \boxed{\dfrac{f''(\xi)\,(x-a)^2}{2!}}$

But $f''(x) = 0 \Big|_{x=\pi} \Longrightarrow f'''(x) = -\cos x \Big|_{x=\pi} = \underline{1}$

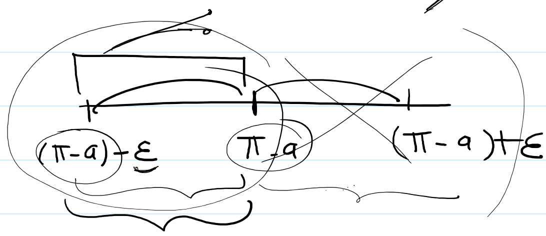
$E(x) = \dfrac{f'''(\xi)}{3!}(x-a)^3$

$|E(x)| < \dfrac{1}{6}(x-a)^3 \Big|_{x=\pi} = \dfrac{1}{6}(\pi - a)^3 = \varepsilon$

ii) $I = \Big((\pi - a) - \varepsilon,\ (\pi - a)\Big)$

$L(\pi)$

$\underbrace{(\pi - a)}$



err = sin(3,14) − Δ

err = $-6{,}733 \times 10^{-10}$

obtained by matlab

err * 3 = $\underline{-2{,}0199 \times 10^{-9}}$

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15) $\sqrt{50}$ $\qquad x = 50, \quad a = 49.$

- Obtain a linear approximation
- Error
- Interval where the exact solution is.

$f(x) = \sqrt{x} = (x)^{1/2}$

1) $f(x) \approx \underbrace{L(x)}_{P_1(x)} = f(a) + \dfrac{f'(a)}{1!}(x-a)^1$ ...

2) $|\text{Error}| = |f(x) - L(x)| < K \underbrace{\dfrac{(x-a)^2}{2!}}$ $\qquad K = \max\limits_{s \in (a-\delta,\, a+\delta)} |f''(s)|$

3) $|f(x) - L(x)| < \varepsilon \Rightarrow L(x) - \varepsilon < f(x) < L(x) + \varepsilon$

Cor A: $f''(t) > 0 \Rightarrow L(x) < f(x)$ $\qquad$ Interval $(L(x),\ L(x)+\varepsilon)$

$f''(t) < 0 \Rightarrow f(x) < L(x)$ $\qquad$ Interval $(L(x)-\varepsilon,\ L(x))$

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad x = 50, \quad a = 49$

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}}$

$f''(x) = -\frac{1}{...} x^{-3/2}$

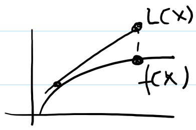
$$f'(x) = \frac{1}{2} \quad = \quad \frac{}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{4} x^{-3/2} = -\frac{1}{4} \frac{1}{\sqrt[2]{x^3}}$$

1) $L(x) = \sqrt{49} + \frac{1}{2}\frac{1}{\sqrt{49}}(x-49)$

$L(50) = 7 + \frac{1}{2}\frac{1}{7}(50-49) = 7 + \frac{1}{2 \cdot 7} = \frac{2 * 7^2 + 1}{14} = \frac{99}{14}$

2) $f''(x) = -\frac{1}{4}\frac{1}{\sqrt[2]{x^3}}$ $\qquad f''(t) < 0$



$|k| = |f''(49)| = \frac{1}{4}\frac{1}{\sqrt{49 \cdot 49^2}} = \frac{1}{4 \cdot 49 \cdot 7} = \frac{1}{49 * 28}$

$|\text{Error}| < \frac{K}{2}(x-a)^2 \Rightarrow |\text{Error}| < \frac{1}{49 * 28} * \frac{1}{2}(\underbrace{50-49}_{1})^2$

$|\text{Error}| < \frac{1}{2744} \simeq 0{,}00036443$

3) $I = \left(\frac{99}{4} - \frac{1}{2744}, \frac{99}{4}\right)$

$(L(x) - \varepsilon \quad , L(x))$

________ , ______________ /

Dugg på Torsdag

→ ε-δ definition for limits

→ $(f^{-1})'$

→ Arc sin
Arc cos
⋮
$\log_a x$, a^x

Example 7 page 281

Obtain Taylor polynomial or order 3 for $f(x) = e^{2x}$ about $a = 1$ from the corresponding Maclaurin polynomial for e^x ($a = 0$)

1) Maclaurin polynomial $a=0$ degree $n=3$.

$\boxed{a=0}$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$$e^x = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3$$

$$\boxed{e^x = 1 + 1(x) + \frac{1}{2!}x^2 + \frac{1}{3!}x^3}$$

2) $g(x) = f(2x) = e^{\boxed{2x}} \quad . \quad a=0$

1

2) $g(x) = f(2x) = e^{\boxed{2x}}$, $a = 0$

$$e^{\boxed{2x}} = 1 + \frac{1}{1!}\boxed{2x} + \frac{1}{2!}\boxed{2x}^2 + \frac{1}{3!}\boxed{2x}^3$$

3) $\boxed{a = 1}$ Change of variable

$\boxed{Z = 2(x-1)}$

$$x = 1 + (x-1)$$

$$e^{2x} = e^{2(1+(x-1))} = e^{2+2(x-1)} = e^2 \cdot e^{2(x-1)} = e^2 \underbrace{e^{\boxed{Z}}}_{*}$$

Now:

$$e^{2x} = e^2\left[1 + \frac{1}{1!}Z + \frac{1}{2!}Z^2 + \frac{1}{3!}Z^3\right] + O(z^4)$$

$$e^{2x} = e^2\left[1 + \frac{1}{1!}(2(x-1)) + \frac{1}{2!}(2(x-1))^2 + \frac{1}{3!}(2(x-1))^3\right] + O\left((2(x-1))^4\right)$$

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§ Evaluation of $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = ?\left[\frac{\infty}{\infty}\right]?$ or $?\left[\frac{0}{0}\right]?$

* 1) Taylor expansion / Taylor approximation of f & g

* 2) L'Hopital Rule

under certain conditions $\lim\limits_{x\to a} \frac{f(x)}{g(x)} \overset{*}{=} \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$

1) $\lim\limits_{x\to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2} = \left[\frac{0-0}{2-2}\right] = \left[\frac{0}{0}\right]$???

$\circledast$ Method: substitute $f(x) = \underbrace{P_n(x)}_{\text{Maclaurin}} + O(x^{n+1})$

$$\sin(x) = 0 + \frac{1}{1!}(x) - \frac{1}{3!}(x)^3 + O(x^5)$$

$$\sin(2x) = 0 + \frac{1}{1!}(2x) - \frac{1}{3!}(2x)^3 + O((2x)^5) = \underbrace{O(2^5x^5) = O(x^5)}$$

$$\sin(2x) = 0 + \frac{1}{1!}(2x) - \frac{1}{3!}(2x)^3 + O((2x)^5) = \underbrace{O(2^5 x^5) = O(x^5)}$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + O(x^4)$$

$$\lim_{x\to 0} \frac{2\left[x - \frac{1}{3!}x^3 + O(x^5)\right] - \left[2x - \frac{1}{3!}(2x)^3 + O(x^5)\right]}{2\left[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + O(x^4)\right] - \left[2 + 2x + 2x^2\right]} =$$

$$\lim_{x\to 0} \frac{\left(-\frac{2}{3!} + \frac{8}{3!}\right)x^3 + O(x^5)}{\frac{2}{6}x^3 + O(x^4)} =$$

$$\lim_{x\to 0} \frac{\left(\frac{6}{6}\right)\cancel{x^3} + \cancel{O(x^5)}}{\left(\frac{2}{6}\right)\cancel{x^3} + \cancel{O(x^4)}} = \underline{\underline{3}} + \cancel{O(x)} \to 0$$

Theo. 3 (§ 4.2) L'Hôpital

(page $\frac{(229)}{(231)}$ upplaga 8 / upp. 9)

f, g differentiable

$g'(x) \neq 0$

$$\lim_{x\to a_+} f(x) = \lim_{x\to a_+} g(x) = 0$$

$$\lim_{x\to a_+} \boxed{\frac{f'(x)}{g'(x)} = L}$$

Then $\lim_{x\to a_+} \frac{f(x)}{g(x)} = L.$

Ex2 $\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{2e^x - 2 - 2x - x^2} = \lim_{x\to 0} \frac{f(x)}{g(x)} \overset{?}{=} \left[\frac{0}{0}\right]$ ($0 = a$)

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$$

L'Hopital Rule ???

L'Hopital Rule

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cancel{2}\cos x - \cancel{2}\cos(2x)}{\cancel{2}e^x - \cancel{2} - \cancel{2}x} = \left[\frac{2-2}{2-2}\right] = \left[\frac{0}{0}\right]$$

L' Hopital again

$$\lim_{x\to 0} \frac{-\sin x + 2\sin(2x)}{e^x - 1} = \left[\frac{0 + 2(0)}{1-1}\right] = \left[\frac{0}{0}\right] ??$$

L' Hopital again

$$\lim_{x\to 0} \frac{-\cos x + 4\cos(2x)}{e^x} = \left[\frac{-1+4}{1}\right] = 3$$

Topics

1) ε-δ formal definition for limit
2) f^{-1} , $(f^{-1})'$
3) $\begin{pmatrix} \text{Arc}\sin x \\ \text{Arc}\cos x \end{pmatrix}'$ 4) $\log_a x$, a^x

$\circledast$ $\begin{cases} x \to a_+ \\ x \to \infty \\ x \to b_- \end{cases}$

$x \to 0$

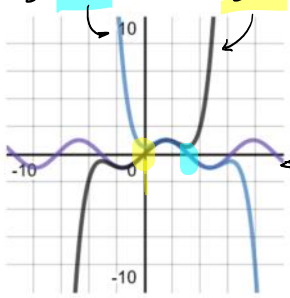
$\longrightarrow$ $\boxed{\left[\frac{0}{0}\right] \text{ or } \left[\frac{\infty}{\infty}\right]}$

L'Hôpital Rule is also valid when $\circledast$

Taylor series / Taylor Polynomial / Taylor Expansions

1) OBS: The Taylor polynomial IS LOCAL. It works fine in a vicinity of one specific $\underline{\underline{a}}$. If you change the point of analysis, everything changes!

$$-(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{x^5}{5!} = P_5(x-\pi) \qquad P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$



← sin(x)

Look this example.

2) OBS: Please, check the change of variables again to obtain the Taylor (MacLaurin) polynomial with respect to different analysis points (a)