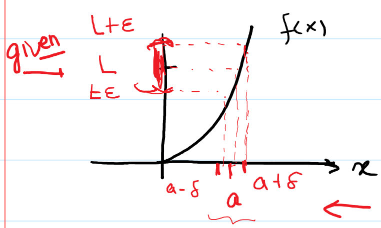
1) a) Write the formal definition of limit

b) Determine the interval where x must be located if $f(x) = x^2$ must be within $\varepsilon = 0{,}1$ distant from $L = 4$.

Lösning:

1.a) $\lim\limits_{x \to a} f(x) = L \iff \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 \;/\; |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$



← you have to find.

1.b) Numerically

given $\varepsilon = 0{,}1$, we need to find $\delta = \delta(\varepsilon)$ (depending on ε)

such that $|x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$.

$$\underbrace{|x^2 - 4| < \varepsilon}_{\text{Image}} \Rightarrow -\varepsilon < x^2 - 4 < \varepsilon \Rightarrow$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon \Rightarrow$$

$$\sqrt{3{,}9} = \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon} = \sqrt{4{,}1} \quad \leftarrow \text{domain}$$

$$1{,}97484176581 < x < 2{,}02484567313$$

$$\boxed{-0{,}0251582 < x - 2 < 0{,}02484567313}$$

So, assuming $\delta \leq \min\{0{,}02484567313, 0{,}0251582\}$ (*)

we can garantie that

$$\underbrace{|x-2| < \delta}_{} \Rightarrow \underbrace{|x^2 - 4|}_{} < \varepsilon.$$

$x \in (2-\delta, 2+\delta)$

with $\delta = 0{,}02 = 2\varepsilon$

for example.

Extra

Just to be sure, we confirm our answer: δ given by (*)

$$\underbrace{|x-2| < \delta}_{\text{start}} \Rightarrow -\delta < x - 2 < \delta$$

$$2 - \delta < x < 2 + \delta$$

$$(2-\delta)^2 < x^2 < (2+\delta)^2$$

Start

Restriction

$$2-\delta < x < 2+\delta$$

$$(2-\delta)^2 < x^2 < (2+\delta)^2$$

$$4-4\delta+\delta^2 < x^2 < 4+4\delta+\delta^2$$

$$-4\delta+\delta^2 < x^2-4 < 4\delta+\delta^2$$

image

$$\boxed{\delta^2+4\delta < \varepsilon}$$

$$\delta^2+4\delta-\varepsilon<0$$

we compute first the roots and then analise "<0"

$\delta = \delta(\varepsilon)$

$$\delta_{1,2} = \frac{-4 \pm \sqrt{16+4\varepsilon}}{2} \Rightarrow \delta = \frac{-4 \pm \cancel{2}\sqrt{4+\varepsilon}}{\cancel{2}} = -2 \pm \sqrt{4+\varepsilon}$$

$$\delta>0 \Rightarrow \quad \delta = -2+\sqrt{4+\varepsilon}$$

$$(\delta^2+4\delta-\varepsilon)<0 \Rightarrow \quad \delta < -2+\sqrt{4+\varepsilon}$$

for the choice $\varepsilon = 0{,}1$, we have

$$\delta < -2+\sqrt{4+0{,}1} = 0{,}0248456731317.$$

So, what we have done was correct!

2. $f(x) = \dfrac{1-2x}{1+x}$

a) $D_f = ?$

b) f is one-to-one?

c) If f^{-1} existis, compute f^{-1} and $D_{f^{-1}}$

d) Use implicit differentiation to obtain $\boxed{(f^{-1})'}$

Lösning:

2.a) $f(x) = \dfrac{1-2x}{1+x}$ $\quad 1+x \neq 0 \Rightarrow x \neq -1$

$$D_f = \mathbb{R} \setminus \{-1\} = (-\infty,-1) \cup (-1,+\infty)$$

2b) f is one-to-one if $f'(x)>0$ or $f'(x)<0$ $\forall x \in (a,b)$

$$f'(x) = \frac{(1-2x)'}{(1+x)} + (1-2x)\cdot(-1)(1+x)^{-2}\cdot 1 =$$

$$f'(x) = \frac{-2}{(1+x)} - \frac{(1-2x)}{(1+x)^2} = \frac{-2(1+x) -1+2x}{(1+x)^2} =$$

$$f'(x) = \frac{-2-\cancel{2x}-1+\cancel{2x}}{(1+x)^2} = \frac{-3}{(1+x)^2} < 0 \quad \forall x \in D_f.$$

Thus f is one-to-one on $D_f = \mathbb{R} \setminus \{-1\}$.

2.c) f one-to-one $\Rightarrow$ f^{-1} exists.

$$\boxed{y = f^{-1}(x)}$$

$\Downarrow$

$$f(y) = x \Rightarrow \frac{1-2y}{1+y} = x \Rightarrow$$

$1-2y = x(1+y) = x+xy \Rightarrow$

$1-2y-xy = x \Rightarrow$

$y(-2-x) = x-1 \Rightarrow$

$$\boxed{y = -\frac{(x-1)}{(x+2)}}$$

$x+2 \neq 0 \Rightarrow x \neq -2.$

$D_{f^{-1}} = \mathbb{R} \setminus \{-2\}$

$$\boxed{\text{So, } f^{-1}(x) = -\frac{(x-1)}{(x+2)}}$$

$$\boxed{D_f = \mathbb{R} \setminus \{-2\}}$$

2.d) Obs we have an expression (an explicit formula) for $f^{-1}(x) = y$, so we could differentiate this formula directly using the derivation rules.

Obs 2. The exercice asks you to use the implicit differentiation.

So: $f(x) = \dfrac{1-2x}{1+x}$

Our goal: compute $\dfrac{d}{dx} f^{-1}(x)$

$\circledast$ $\boxed{x = f(f^{-1}(x))}$

Implicit Differentiation

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$1 = \dfrac{dx}{dx} = f'(f^{-1}(x)) \cdot \dfrac{d}{dx} f^{-1}(x)$

Chain Rule

So: $\dfrac{d}{dx} f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{\boxed{f'(y)}}$

Thus $f(y) = \dfrac{1-2y}{1+y}$

$f'(y) = \dfrac{d f(y)}{dy} \overset{\circledast}{\equiv} \dfrac{-3}{(1+y)^2}$

we already have this information, because we testes if $f(x)$ is 1:1 so we computed $f'(x)$.

Now $f'(y)$ is the same expression but with y as variable instead!

So $\boxed{\dfrac{d}{dx} f^{-1}(x) = \dfrac{1}{\left(\dfrac{-3}{1+y^2}\right)}}$

$$\boxed{\frac{d}{dx} f^{-1}(x) = -\frac{(1+y)^2}{3} \qquad (i)}$$

implicit formulation

Obs $y = f^{-1}(x) = -\dfrac{(x-1)}{(x+2)}$

if we substitute $y = f^{-1}(x)$ in (i), we have

$$-\frac{1}{3}\left(1 - \frac{(x-1)}{(x+2)}\right)^2 =$$

$$-\frac{1}{3}\left(\frac{\cancel{x}+2-\cancel{x}+1}{(x+2)}\right)^2 =$$

$$-\frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{(x+2)}\right)^2 = \boxed{-\frac{3}{(x+2)^2}}$$

$$\boxed{\frac{d}{dx} f^{-1}(x) = -\frac{3}{(x+2)^2}}$$ explicit formula

Just to check:

$y = f^{-1}(x) = -\frac{(x-1)}{(x+2)}$

We could also $\frac{d}{dx}\left(-\frac{(x-1)}{(x+2)}\right) = -\left[\frac{1}{(x+2)} + (x-1)(-1)(x+2)^{-2}\cdot 1\right]$

$$= -\left[\frac{1}{x+2} - \frac{x-1}{(x+2)^2}\right] = -\left[\frac{\cancel{x}+2-\cancel{x}+1}{(x+2)^2}\right] =$$

$$= \boxed{-\frac{3}{(x+2)^2}}$$, confirming the explicit formula obtained before.

this was just to check!

3) $\frac{d}{dx} f(x)$, $f(x) = x^{\sqrt{x}}$ =

Here we have to use logarithmic differentiation page 184 § 3.3

$y = x^{\sqrt{x}} = f(x)^{g(x)}$

properties

$\ln(y) = \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\,\ln x$

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(\sqrt{x}\cdot\ln x\right)$$

$$\underbrace{\frac{1}{y}\cdot\frac{dy}{dx}} = \frac{1}{2}(x)^{-1/2}\cdot\ln x + \sqrt{x}\,\frac{1}{x}$$

$$\frac{dy}{dx} = y\left(\frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\,\frac{1}{x}\right)$$

$$\underbrace{\underbrace{x^{1/2}\cdot x^{-1}}_{x^{-1/2}}}_{\frac{1}{\sqrt{x}}}$$

$$\frac{dy}{dx} = \underbrace{x}^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}}\right) =$$

$$\boxed{\frac{dy}{dx} = x^{\sqrt{x}}\,\frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right)}$$