Analytic Geometry & Linear Algebra

Adams Chapter 10
§10.1 – §10.2 / §10.3, §10.4

Lay Chapter 1 / §L1.1, §L1.2

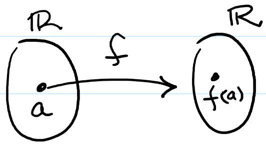
Overview

Adams / page 569.

1) Types of functions studied in Calculus.

1.1 $f: \mathbb{R} \to \mathbb{R}$

$x \to f(x)$

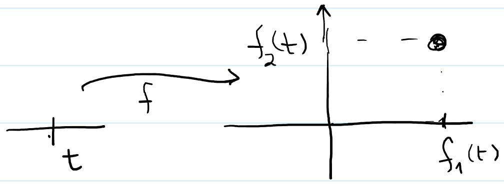


Real valued function $f(x)$ of a single variable x

Chapters 1–9 Adams

1.2 $f: \mathbb{R} \to \mathbb{R}^n$

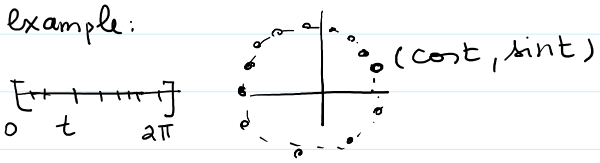
$t \to (f_1(t), f_2(t), \dots, f_n(t))$



Vector valued function $(f_1(t), f_2(t), \dots)$ of a single variable t

Chapter 11 Adams

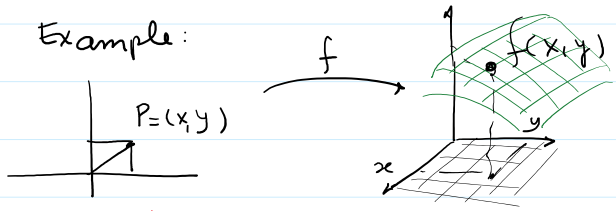
example:



1.3 $f: \mathbb{R}^n \to \mathbb{R}$

$\underbrace{(x_1, x_2, \dots x_n)}_{\vec{x}} \to f(x_1, x_2, \dots, x_n)$

Example:



Chapters 12–14 Adams

Real valued function of a real vector variable $\vec{x} = (x_1, x_2, \dots x_n)$

1.4 $f: \mathbb{R}^n \to \mathbb{R}^m$

$(x_1, \dots x_n) \to \left(f_1(x_1 \dots x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots x_n)\right)$

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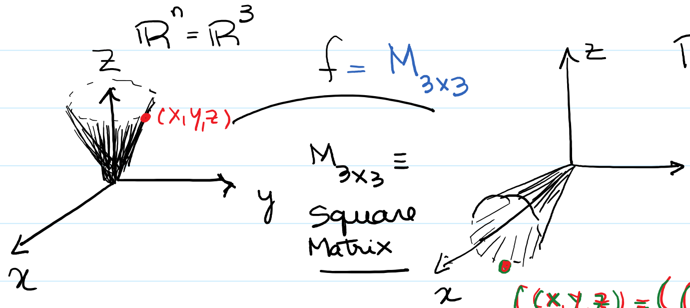
Example

$\mathbb{R}^n = \mathbb{R}^3$ $\quad f = M_{3\times3}$ $\quad \mathbb{R}^m = \mathbb{R}^3$

$M_{3\times3}$ = Square Matrix



$f(x,y,z) = \left(f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)\right)$

$x \longrightarrow x$ $\quad f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

$\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\qquad \vec{w} = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

vector valued function of a vector valued variable

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{f} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

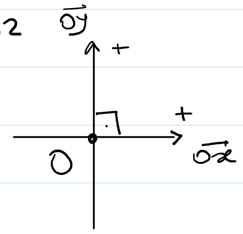
$$f(\bar{x}) = \bar{w}.$$

Obs: Vi ska använda alt om vector teori att lösa / förstår den där typ av problem / fråga ((1.4)).

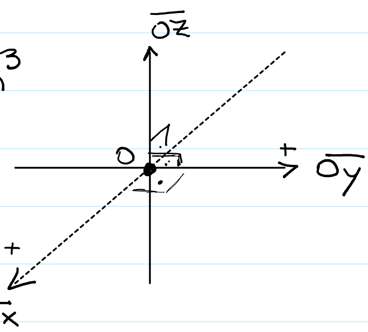
§10.1 Analytic Geometry in 3-dimensions. Adams pag 570

1. Cartesian Coordinate System.
   → 1 starting point called Origin
   → Mutually perpendicular axis.

$\mathbb{R}^2$



$\mathbb{R}^3$



$O = (0,0,0)$

$\vec{Ox} \perp \vec{Oy}$

$\sphericalangle(\vec{Ox}, \vec{Oy}) = 90°$

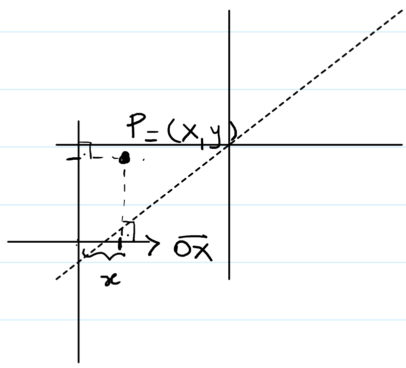
$O = (0,0)$

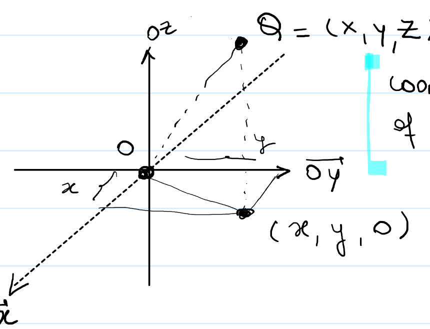
$\vec{Ox} \perp \vec{Oy} \quad \therefore \sphericalangle(\vec{Ox}, \vec{Oy}) = 90°$

$\vec{Ox} \perp \vec{Oz} \quad \therefore \sphericalangle(\vec{Ox}, \vec{Oz}) = 90°$

$\vec{Oy} \perp \vec{Oz} \quad \therefore \sphericalangle(\vec{Oy}, \vec{Oz}) = 90°$

1.1 Coordinates in $\mathbb{R}^2$





Coordinates of points in $\mathbb{R}^3$

Coordinates of Points in $\mathbb{R}^n$ $\quad P = (x_1, x_2, x_3, x_4, \ldots, x_{n-1}, x_n)$

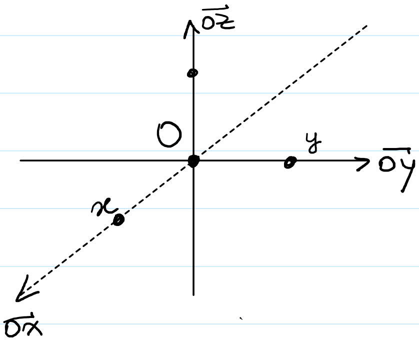
$n \in \mathbb{N}$

Example: Våra personnummer $\quad$ XXXX MM DD – abcd =

$n \in \mathbb{N}$.

Example: Våra personnummer $XXXX\ MM\ DD - abcd = (x_1, x_2, x_3, x_4, x_5\, x_6, x_7\, x_8\ x_9\, x_{10}\, x_{11}\, x_{12}) \in \mathbb{R}^{12}$.

1.3) Equations



free parameters

$P \in \vec{Ox} \Rightarrow P = (x, 0, 0) = x(1,0,0),\ x \in \mathbb{R}$

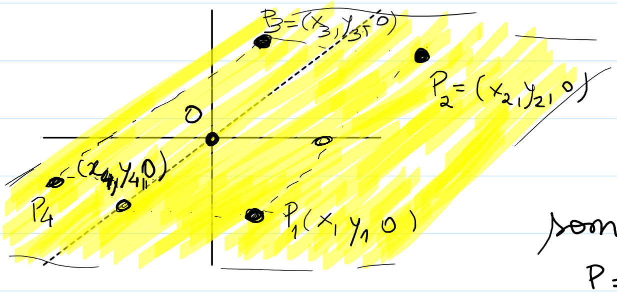
$Q \in \vec{Oy} \Rightarrow Q = (0, y, 0) = y(0,1,0),\ y \in \mathbb{R}$

$T \in \vec{Oz} \Rightarrow T = (0,0,z) = z(0,0,1),\ z \in \mathbb{R}$

Equations of the Coordinate Axis.

1) the axis $\vec{Ox}$ has equation: $y = 0,\ z = 0$ and $x \in \mathbb{R}$.
2) the axis $\vec{Oy}$ has equation $x = 0,\ z = 0$ and $y \in \mathbb{R}$
3) the axis $\vec{Oz}$ has equation $x = 0,\ y = 0$ and $z \in \mathbb{R}$.

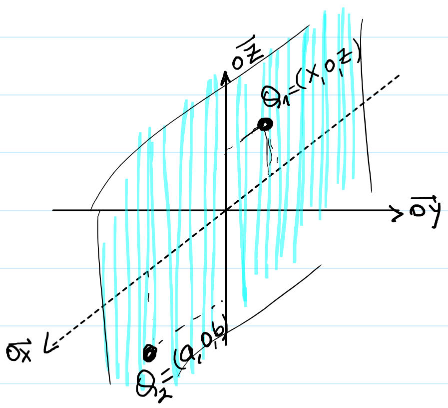
Equations of the Coordinate Planes.



Π_{xy} = plane xy

all points $P \in \Pi_{xy}$ have something in common. $\boxed{z = 0.}$

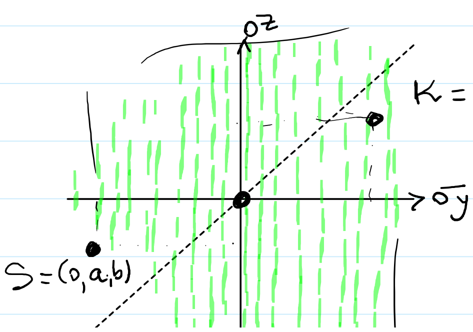
$P = (x, y, 0)$:



Π_{xz} = plane xz

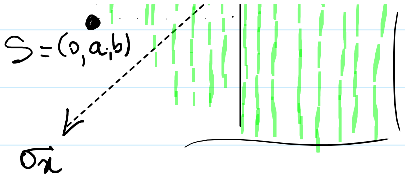
all points $P \in \Pi_{xz}$ have something in common: $\boxed{y = 0}$

$Q = (x, 0, z)$



Π_{yz} = plane yz

all points $K \in \Pi_{yz}$ has one property in common: $\boxed{x = 0}$



comon: $\boxed{X=0}$

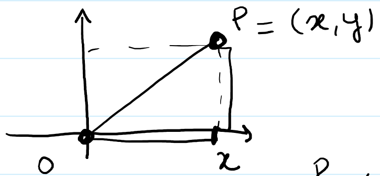
$K=(0,y,z)$

1) The equation of Π_{xy}: $z=0$ $x\in\mathbb{R},\ y\in\mathbb{R}$
2) The equation of Π_{xz}: $y=0$ $x\in\mathbb{R},\ z\in\mathbb{R}$
3) The equation of Π_{yz}: $x=0$, $y\in\mathbb{R},\ z\in\mathbb{R}$.

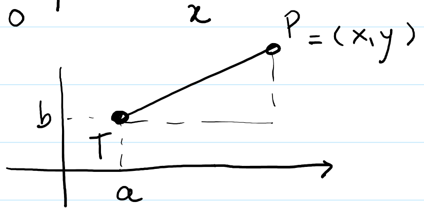
Obs: In $\mathbb{R}^n$, a hyperplane is a $(n-1)$-dimensional set of points in $\mathbb{R}^n$ that satisfy the equation $x_k=0$, $k=1,2,\dots,n$.

2. Distance between 2 Points in Space ($\mathbb{R}^3$) & in $\mathbb{R}^n$.

2.1

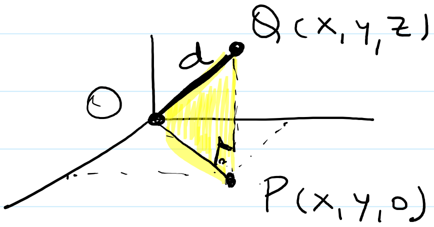


$$d(P,O)=\sqrt{(x-0)^2+(y-0)^2}$$
$$d(P,O)=\sqrt{x^2+y^2}$$



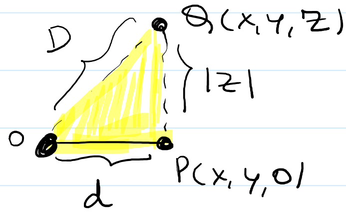
$$d(P,T)=\sqrt{(x-a)^2+(y-b)^2}$$

2.2) $\mathbb{R}^3$



$$d(P,O)=\sqrt{(x-0)^2+(y-0)^2+(0-0)^2}=\sqrt{(x-0)^2+(y-0)^2}=d$$

Obs: The Triangle



$$D=d(Q,O)=$$
$$=\sqrt{d^2+(z-0)^2}=$$
$$=\sqrt{\underbrace{(x-0)^2+(y-0)^2}_{d}+\underbrace{(z-0)^2}_{z^2}}$$

General Case : $P, Q \in \mathbb{R}$ $\quad P = (x, y, z)$, $Q = (a, b, c)$

$$d(P,Q) = d(Q,P) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

General Case II , $P = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$

$Q = (a_1 a_2 \dots a_n) \in \mathbb{R}^n$

$$d(P,Q) = \sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2}$$

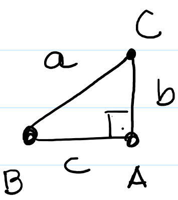
$$= \sqrt{\sum_{k=1}^{n} (x_k - a_k)^2}$$

n = dimension of the space $\mathbb{R}^n$.

Example 1 – page 570.

Show that the triangle ABC, with extreme points given as: $A = (1, -1, 2)$, $B = (3, 3, 8)$ & $C = (2, 0, 1)$ has a right angle.

Lösning:



$|BC| = d(B,C) = \sqrt{59} = a$

$|AC| = d(A,C) = \sqrt{3} = b$

$|AB| = d(A,B) = \sqrt{56} = c$

If ABC has a right angle, then the Pitagoras Theorem must be valid:

$$a^2 = b^2 + c^2$$

$$(\sqrt{59})^2 = (\sqrt{3})^2 + (\sqrt{56})^2 \Rightarrow$$

$$59 = 3 + 56 \quad \checkmark$$

So, ABC has a right angle.

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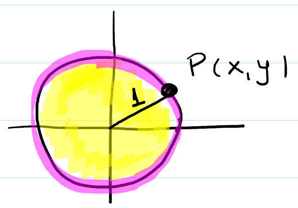
3) More on Equations

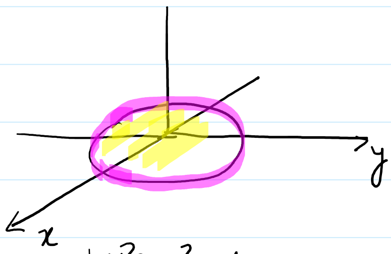
3.1) In $\mathbb{R}^2$



$x^2 + y^2 = 1$: circle of Radius 1

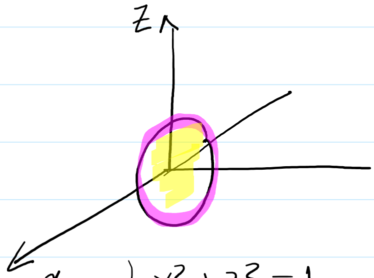
$x^2 + y^2 < 1$ interior of the circle

circle



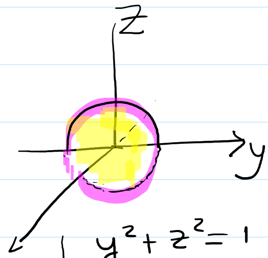
$\begin{cases} x^2+y^2=1 \\ z=0 \end{cases}$

or $\begin{cases} x^2+y^2<1 \\ z=0 \end{cases}$



$\begin{cases} x^2+z^2=1 \\ y=0 \end{cases}$

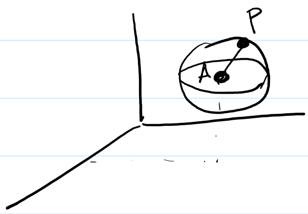
OR $\begin{cases} x^2+z^2<1 \\ y=0 \end{cases}$



$\begin{cases} y^2+z^2=1 \\ x=0 \end{cases}$

OR $\begin{cases} y^2+z^2<1 \\ x=0 \end{cases}$

$d(A,P)=r$, r given as a fixed radius
A given as a reference point



$d(A,P)=r$

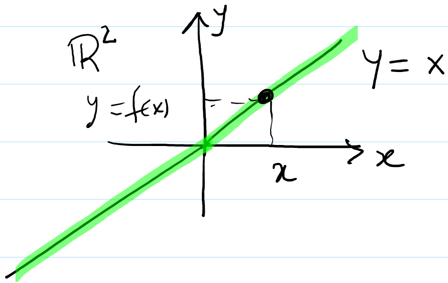
$A=(a,b,c)$
$P=(x,y,z)$

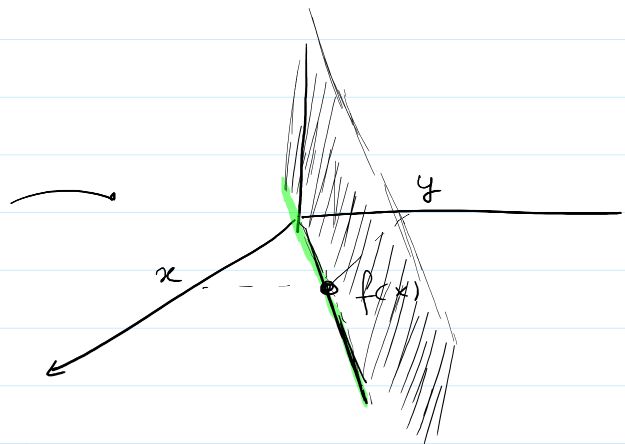
$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r$$

Eq of the circle in $\mathbb{R}^3$ centered on $A=(a,b,c)$ with radius r.

$$\boxed{(x-a)^2+(y-b)^2+(z-c)^2=0}$$

3.2) $\mathbb{R}^2$



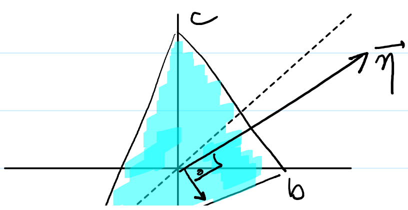


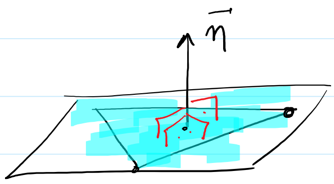
Equation of a plane in different formulations.

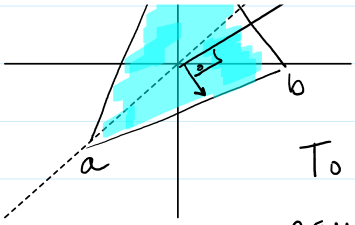
$P=\{(x,y,z)\in\mathbb{R}^3 \,/\, y=x \wedge z\in\mathbb{R}\}$

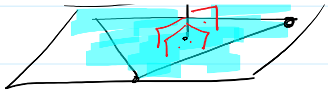
$P=\{(x,x,z) \,/\, x\in\mathbb{R} \;\&\; z\in\mathbb{R}\}$

$P=\{x(1,1,0)+z(0,0,1),\ x\in\mathbb{R},\ z\in\mathbb{R}\}$







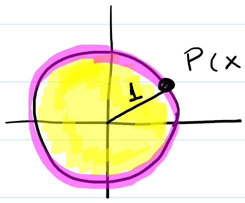


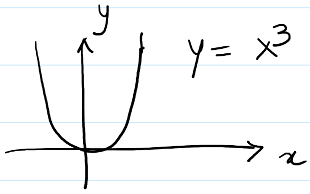
To understand how to define the equation of a plane in $\mathbb{R}^3$ (or $\mathbb{R}^n$), we will need to understand & know how to compete:

$\vec{n}$ = the direction that is orthogonal to all segments contained on the plane.

3.3.

In $\mathbb{R}^2$

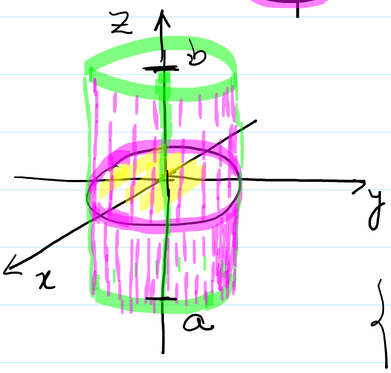




$y = x^3$

In $\mathbb{R}^3$

$\begin{cases} x^2+y^2=1 \\ \cancel{z=0} \end{cases}$

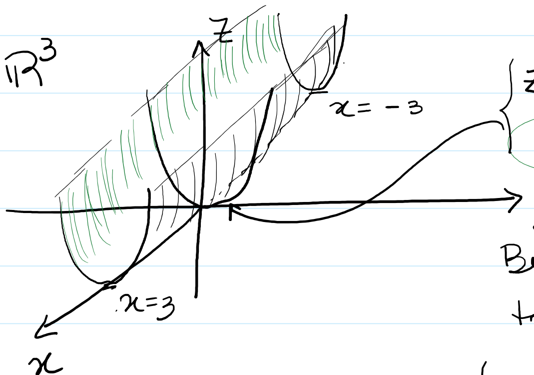


What happens If:

a) instead of $z=0$, Now we allow $z \in [a,b]$?

$\begin{cases} x^2+y^2=1 \\ a \le z \le b \end{cases}$

In $\mathbb{R}^3$



$\begin{cases} z = y^2 \\ x=0 \end{cases}$

But If we allow z to variate, then we have

$\begin{cases} z = y^2 \\ -3 \le x \le 3 \end{cases}$

Therefore, we are going to have the possibility to define many other regions on $\mathbb{R}^3$. / on $\mathbb{R}^n$.

$\mathbb{R}^n$ space:

$$\mathbb{R}^3 = \{(x,y,z) \,/\, x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}$$
$$\{(x,y,z) \,/\, (x,y,z) = (x,0,0) + (0,y,0) + (0,0,z)\}$$
$$\{(x,y,z) \,/\, (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)\}$$

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \,/\, x_i \in \mathbb{R}^i\}$$
$$= \{(x_1, x_2, \dots x_n) \,/\, (x_1 \dots x_n) = x_1(1, \dots 0) + \dots + x_n(0, \dots, 0, 1)\}$$

$\mathbb{R}$: (———) $\equiv (a,b)$ open interval

$[\text{———}]$ $\equiv [a,b]$ closed interval

$\mathbb{R}^2$

$d(P,C) = r$ The boundary of a Ball

$d(P,C) < r$ The interior of a Ball

$d(P,C) \leq r$ A ball with boundary

So we are going to call

$d(P,C) < r$ an open ball in $\mathbb{R}^2$ / $\mathbb{R}^n$

$d(P,C) \leq$ an closed ball in $\mathbb{R}^2$/ $\mathbb{R}^n$

$d(P,C) = r$ are called the boundary of the ball (in $\mathbb{R}^2/\mathbb{R}^n$)

- → x_1 & x_2 are called interior points
- → y is called exterior point
- → t is an element on the boundary.

Vicinity of C is any open Ball centered on C with some chosen distance r.

$$V_C = B_r(C) = \text{ball centered on } C \text{ with radius } r.$$

§ 10.2 Vectors on $\mathbb{R}^n$.

Obs: In problems involving Analitik Geometry we are focusing on $\mathbb{R}^2$ & $\mathbb{R}^3$.

In problems involving Calculus & Linear Algebra, we might study more applications in $\mathbb{R}^n$.

As we are going to see more rigorously, $\mathbb{R}^2$, $\mathbb{R}^3$ and all $\mathbb{R}^n$ $n>3$ all vectorial spaces and share the same structure.

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1) Definition: (Vector) page 575.

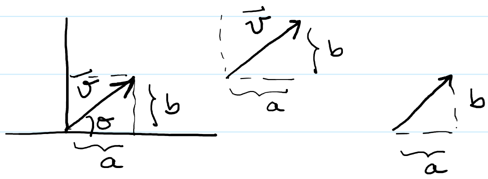
$\vec{v}$ = vector is a quantity that involves:

- $\|\vec{v}\|$ = magnitude (size or length) of $\vec{v}$
- direction of $\vec{v}$ = slope of $\vec{v}$.
- orientation: $-\vec{v}$ or $+\vec{v}$.

in $\mathbb{R}^2$ & $\mathbb{R}^3$ these notions can have geometrical interpretations, while in $\mathbb{R}^n$ $n>3$ these concepts are treated in a more analytical way.) since the geometrical interpretations can be lost.

$\mathbb{R}^2$

$\vec{v} = (a,b)$



$\|\vec{v}\| = \sqrt{a^2+b^2}$

slope $= \tan\theta = \dfrac{b}{a}$

$\vec{v} = (1,1)$ for example:

Question: $\underbrace{\vec{v} = (1,1)}_{\text{vector}} \overset{?}{=} \underbrace{P = (1,1)}_{\text{point}}$

$\vec{v}$ = can be treated as an oriented segment

$\vec{v} = \overrightarrow{AB} = B - A$

$= (3,0) - (2,-1) =$

$= (3-2, 0+1)$

B = final Point. (x_b, y_b)

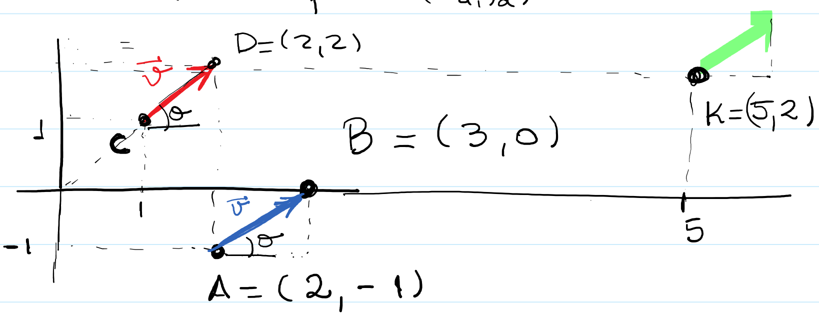
A = initial point (x_a, y_a)

$D = (2,2)$

$= (3-2, 0+1)$

$= (1, 1)$

$A =$ initial point (x_a, y_a)



But

$\vec{v} = \vec{CD} = D - C$

$= (2,2) - (1,1) =$

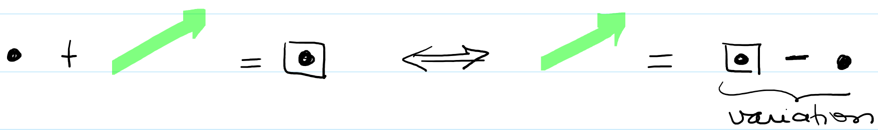
$\vec{v} = (1,1)$

och $\vec{v} = \vec{OC} = C - O = (1,1) - (0,0)$ ok!

$= (1,1)$

Alla point can be associate to a vector directly.

2) Sum : Point + Vector = Point.



$A + \vec{v} = B \iff \vec{v} = B - A = \underbrace{\vec{AB}}_{\text{oriented segment.}}$

$\vec{AB} = B - A = (3-2, 0+1) = (1,1)$

Slope $= \tan\vartheta = \dfrac{3-2}{0+1} = \dfrac{1}{1} = 1 \Rightarrow \underline{\vartheta = 45^\circ}$

3) Operations $\qquad \vec{u} = (x_u, y_u) \quad \vec{v} = (x_v, y_v)$

$(+) \quad \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(\vec{u}, \vec{v}) \longrightarrow \vec{u} + \vec{v} = (x_u + x_v, y_u + y_v)$

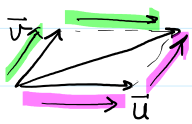
$\cdot : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(\lambda, \vec{u}) \longrightarrow \lambda \cdot \vec{u}$

$\lambda\vec{u} = \lambda(x_u, y_u) =$

$= (\lambda x_u, \lambda y_u)$

Obs: you can extendend these two operations for $\mathbb{R}^n$, $n \geq 3$.

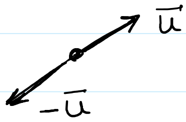


$\vec{u} + \vec{v} = \vec{v} + u$

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

  $\vec{0} =$ null vector of the Space
- given a vetor $\vec{u}$, there exists a vector $-\vec{u}$ such

- given a vetor $\vec{u}$, there exists a vector $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$



$\vec{u} + (-\vec{u}) = \vec{0}$.

- $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$

- $(\lambda+\mu)\vec{u} = \lambda\vec{u} + \mu\vec{u}$

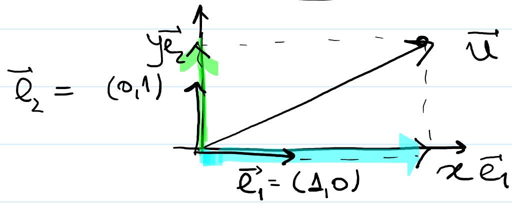
- $\lambda(\vec{u}+\vec{v}) = \lambda\vec{u} + \lambda\vec{v}$

- $(\alpha\beta)\vec{u} = \alpha(\beta\vec{u}) = (\beta\alpha)\vec{u}$.

$\mathbb{R}^2 = \{(x,y) \,/\, x\in\mathbb{R},\ y\in\mathbb{R}\}$

$\{\vec{u}=(x,y) \,/\, \vec{u} = (x,0) + (0,y)\} =$

$\{\vec{u}=(x,y) \,/\, \underline{\vec{u} = x(1,0) + y(0,1)}\}$ ⊛



$\|\vec{e}_1\| = \sqrt{1^2+0^2} = 1$ $\quad$ $\vec{e}_1$ is one unitary vector on the axis $\overrightarrow{ox}$.

$\|\vec{e}_2\| = \sqrt{0^2+1^2} = \sqrt{1} = 1$ $\quad$ $\vec{e}_2$ is another unitary vector, now on the $\overrightarrow{oy}$ axis.

So, we will define it more properly in the next lectures, but $\{\vec{e}_1, \vec{e}_2\}$ can be considered as a generator of all vectors on $\mathbb{R}^2$. And in fact, they are:

Given $\vec{w} = (\alpha,\beta) \in \mathbb{R}^2$, $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$,

it is allways possible to write $\vec{w} = (\alpha,0) + (0,\beta) \Rightarrow$
$\vec{w} = \alpha(1,0) + \beta(0,1)$

This concept can be extended to $\mathbb{R}^n$, with any choice of $n\in\mathbb{N}$.

$$\mathbb{R}^n = \{(x_1, x_2, x_3, \dots, x_n) \;/\; x_i \in \mathbb{R}\} =$$

$$\{(x_1, \dots, x_n) = (x_1, 0 \dots 0) + (0, x_2, \dots 0) + \dots (0, \dots 0, x_n)\} =$$

$$\{\underbrace{\overbrace{(x_1, \dots, x_n)}^{\vec{u}}} = x_1(1, 0 \dots 0) + x_2(0, 1, 0, \dots 0) + \dots + x_n(0, \dots, 0, 1)\}$$

$$\vec{e}_1 = (1, 0, \dots 0) \qquad \vec{e}_2 = (0, 1, \dots 0) \quad \dots \quad \vec{e}_n = (0, 0 \dots 0, 1)$$

$\vec{u}$ is called a linear combination of $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ page 577

We call $\{\bar{e}_1, \vec{e}_2, \dots \bar{e}_n\}$ as the canonical Base of $\mathbb{R}^n$. page 577
Standard

4) The Dot Product. page 581

$\bullet : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ / $\bullet \; \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ / $\bullet \; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$(\bar{u}, \bar{v}) \longmapsto \vec{u} \bullet \bar{v} = \alpha \in \mathbb{R}$ $\qquad (\bar{u}, \bar{v}) \longrightarrow \bar{u} \bullet \bar{v}$ $\qquad (\bar{u}, \bar{v}) \longmapsto \vec{u} \bullet \vec{v}$

Considering $\bar{u} = (x_1, x_2)$ $\bar{v} = (a_1, a_2) \in \mathbb{R}^2$, we have

$$\vec{u} \bullet \bar{v} = (x_1, x_2) \bullet (a_1, a_2) = x_1 \cdot a_1 + x_2 \cdot a_2$$

Analogously for $\mathbb{R}^3$ & $\mathbb{R}^n$.

$$\begin{cases} \bar{u} \in \mathbb{R}^3 \\ \bar{v} \in \mathbb{R}^3 \end{cases} \Rightarrow \vec{u} \cdot \bar{v} = (x_1, x_2 x_3) \bullet (a_1 a_2 a_3) = x_1 \cdot a_1 + x_2 \cdot a_2 + x_3 \cdot a_3$$

$$\begin{cases} \vec{u} \in \mathbb{R}^n \\ \bar{v} \in \mathbb{R}^n \end{cases} \Rightarrow \vec{u} \bullet \vec{v} = (x_1, \dots x_n) \bullet (a_1 \dots a_n) = \sum_{k=1}^{n} x_k \cdot a_k =$$

$$= x_1 a_1 + x_2 a_2 + \dots + x_n a_n.$$

Properties of $\vec{u} \bullet \vec{v}$ page 581 Adams

1) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

2) $\vec{u} \bullet (\vec{v} + \vec{\omega}) = \vec{u} \bullet \vec{v} + \vec{u} \cdot w$

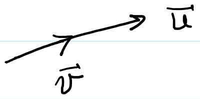
3) $(\lambda \vec{u}) \bullet \vec{v} = \lambda(\vec{u} \bullet \vec{v})$

4) $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2 = |\vec{u}|^2$

Theorem 1 page 581 $(\mathbb{R}^2 / \mathbb{R}^3)$

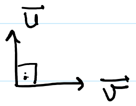
If θ is the angle between $\vec{u}$ and $\vec{v}$ $(0 < \theta < \pi)$,
then $\underline{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$

In particular

a) 

$\vec{u} \parallel \vec{v} \Rightarrow \theta = \measuredangle(\vec{u}, \vec{v}) = \bar{0}$

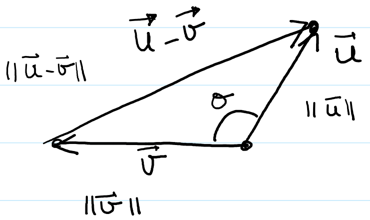
In this case $\cos(\theta) = \cos(0) = 1$.

b) 

$\vec{u} \perp \vec{v} \Leftrightarrow \measuredangle(u, v) = 90^\circ$

och $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos 90^\circ = 0$

Proof of the Theorem: (sida 582)



From the cosine law applied to the triangle with sides $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{u} - \vec{v}\|$

$$\boxed{\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta} \quad (A)$$

on the other Hand, by the properties we know

$$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - \underbrace{\vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u}}_{\text{property (1)}} + \vec{v} \cdot \vec{v}$$

$$\underline{= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v}} \quad (B)$$

Comparing (A) with (B), we have:

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\|\vec{v}\|\cos\theta = \|\vec{u} - \vec{v}\|^2 = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\vec{u} \cdot \vec{v}$$

Thus $\cancel{2}\,\vec{u} \cdot \vec{v} = \cancel{2}\,\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$